Optimal Auction Design in a Common Value Model*

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Abstract

We study auction design when bidders have a pure common value equal to the maximum of their independent signals. In the revenue maximizing mechanism, each bidder makes a payment that is independent of his signal and the allocation discriminates in favor of bidders with *lower* signals. We provide a necessary and sufficient condition under which the optimal mechanism reduces to a posted price under which all bidders are equally likely to get the good. This model of pure common values can equivalently be interpreted as model of resale: the bidders have independent private values at the auction stage, and the winner of the auction can make a take-it-or-leave-it-offer in the secondary market under complete information.

KEYWORDS: Optimal auction, common values, revenue maximization, revenue equivalence, first-price auction, second-price auction, resale, posted price, maximum value game, wallet game, descending auction, local incentive constraints, global incentive constraints.

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1 Introduction

We study auction design when bidders have a pure common value. Each of N bidders receives an independent signal and the pure common value of the object is given by the maximum of the N independent signals of the bidders. We call this environment the "maximum of independent signals" common value model, or a bit shorter, maximum common value model.[1] We have two interpretations in mind. First, each bidder's independent signal – his type – may represent a possible use of the good that the bidder has discovered. Whoever wins the good at the auction will ultimately discover the best possible use, so that lower signals (inferior discoveries) contain no information about the value conditional on the highest signal (best discovery).[2] Second, each signal may represent the bidder's private value of the good, but there is a re-sale market where the good is allocated to the bidder with the highest private value. A bidder's private value gives a lower bound on the highest possible value, but no other information.

We characterize the revenue maximizing mechanism in this environment. Maximum revenue is achieved by a constant – signal independent – participation fee, and a constant – again signal independent – probability of receiving the good. The optimality of constant participation fees and assignment probabilities is valid for all independent and symmetric distributions.

The good may not be allocated in the optimal auction. We obtain necessary and sufficient conditions under which the object is assigned with probability one in the auction. When these conditions are met, the optimal mechanism reduces to a simple posted price mechanism. The optimal posted price is equal to the conditional expectation that the lowest possible signal has about the value of the object. In turn, every bidder is willing to buy the object at the posted price and all bidders are equally likely to receive the good in equilibrium. The necessary and sufficient condition for such an inclusive posted price to be optimal is given by a generalization of the virtual utility formula. The condition essentially requires that the distribution of the value does not put too much mass close to the seller's value for the good, and in particular it requires that the lowest possible value of each bidder is bounded away from the seller's value.

We now describe how we obtain our results. We first argue by contradiction that the optimal auction cannot be characterized by local incentive constraints alone. If it were, the revenue equivalence formula would indicate that the optimal allocation would only allocate the good to bidders who do not have the highest signal. The virtual utility function represents

[1] Bulow and Klemperer (2002) studied the winner's curse in an ascending auction with this common value model, which they described as the "maximum game."

[2] This is consistent with a mineral rights interpretation in Bulow and Klemperer (2002).

value minus the information rents due to binding local downward incentive constraints. In a common value setting, the value of the object is the same for all bidders, and the bidders only differ in their information rent. Now, when the value of the object is the maximum of all signals, an information rent only accrues to the highest type. The information held by all other bidders is locally without influence on the value of the object. But this means that the virtual utility is lowest for the highest type, and higher (and identically so) for all lower types. Thus, the analysis of the local incentive constraints would suggest that the object should be assigned to one of the bidders with low signals, and never assigned to the bidder with the highest signal. Such an allocation would, however, violate global incentive constraints, in that high types would prefer to misreport themselves as lower types.

In fact, we show that the optimal allocation makes bidders indifferent between reporting their true types and reporting *any* lower type. For characterizing optimal revenue, however, it is sufficient to focus on a relatively small collection of downward incentive constraints of the following form: instead of reporting their true type, a bidder could misreport a lower type which is randomly drawn from the prior distribution censored at the bidder's true value. We derive maximum revenue when the mechanism only has to deter local deviations and this one-dimensional family of global deviations, which is necessarily an upper bound on optimal revenue.

The optimal auction resolves this tension between local and global constraints to the maximal extent possible. Namely, it assigns the object as frequently as possible to the low signals, and as infrequently as allowed by global incentive constraints to the high signal. The resulting allocation assigns the object so that each type is *equally* likely to receive the good in an interim sense. This brief description should indicate that the arguments that support the construction of the optimal auction will differ significantly from the standard construction that extends the local incentive constraints to global incentive constraints. Instead, we directly consider a small set of global deviations that will be necessary as well as sufficient to characterize maximum revenue.

We construct an incentive compatible mechanism that exactly achieves the upper bound. In the direct mechanism, all types are asked to make a fixed payment, a participation fee, that is independent of their type. No transfers beyond the participation fee are collected. In terms of the allocation, every type has the same interim expected probability of being allocated the good. These two features of the optimal mechanism resemble a posted price mechanism. However, unlike posted prices, the object is only allocated if the highest realized signal among the bidders exceeds a threshold value. Thus, typically, the probability that the object is assigned to *some* bidder is strictly smaller than one. The second feature distinct from posted prices is that the optimal mechanism discriminates against bidders with higher signals. That is, conditional on the entire signal profile, the optimal mechanism allocates the good to lower types with greater frequency than would a purely random allocation. From an interim point of view of each bidder, given his signal, the conditional probability that the good is allocated to somebody is increasing with the highest signal. Interestingly, the lower probability of receiving the good conditional on a high signal is *exactly balanced* out by the higher interim probability that the good is assigned at all in such a way that the interim probability of receiving the good is constant. We also exhibit an indirect implementation of the optimal mechanism by means of a descending auction with an entry fee.

A fundamental question in the theory of mechanism design is: how should selling mechanisms be structured in order to extract as much revenue as possible from the bidders? Since Myerson's (1981) paper on optimal auctions this problem is typically approached with two essential observations. First, the *revelation principle* says that it is without loss of generality to restrict attention to a class of "direct" mechanisms in which bidders simply report their private information to the mechanism. Second, when private information is independent across bidders and when preferences can be suitably ordered by type, the *revenue equivalence theorem* says that the revenue from a mechanism is determined by the allocation that it induces and the utility of the lowest type. The reason is that there is a unique transfer schedule under which truthful reporting is *locally* optimal for every type. If bidders' values are also additively separable between their own type and others' types—e.g., as in the independent private value (IPV) case—then there is a simple monotonicity condition that characterizes the allocations for which truthful reporting is also *globally* optimal. These tools reduce the problem of maximizing revenue over mechanisms to the problem of maximizing revenue over monotonic allocations, which in the separable case can be essentially solved in closed form.

Since that seminal work there have been remarkably few results that generalize the theory of optimal auctions beyond the private value case. An important generalization of the revenue equivalence result was obtained by Bulow and Klemperer (1996) to a model with interdependent values in which the values are weakly increasing and possibly not additively separable. Both the revelation principle and the revenue equivalence theorem generalize. Importantly, now the transfers no longer depend just on the allocation, but rather on the allocation weighted by the sensitivity of the value to the bidder's private information. In this more general model, however, there is no simple analogue of the monotonicity condition anymore to guarantee the global incentive compatibility of the mechanism. The literature, most notably Bulow and Klemperer (1996), has identified special cases in which the local constraints are sufficient to guarantee the global incentive constraints. When the values are common, this occurs when the information rent is (weakly) smaller for bidders with higher signals, where the information rent is the product of the inverse hazard rate and the sensitivity of the value to the bidder's information. For example, this is the case when the common value is equal to the sum of the bidders' signals[3] and when the distribution of the signals satisfies an increasing hazard rate condition. In this case the value is equally sensitive to all of the bidders' signals, and the increasing hazard rate implies that information rents are smaller for higher types. As a result, it is optimal to bias the allocation in favor of bidders with higher signals. To our knowledge, this is the first paper to extend the theory of optimal auctions in common value environments beyond the case of decreasing information rents.

When the lowest valuation in the support of the distribution is sufficiently greater than the seller's value, then it is not worthwhile to discriminate, and the optimal mechanism reduces to a posted price at which every bidder is willing to purchase the object independent of his signal realization. This relates to an observation of Bulow and Klemperer (2002) that a fully inclusive posted price (i.e., a price at which all types would be willing to purchase the good) generates more revenue than a second-price auction in the maximum common value model. The reason is that in the second-price auction the good is allocated to the bidder with the highest signal, who has the highest information rent, whereas all types are equally likely to be allocated the good under an inclusive posted price, see also Harstad and Bordley (1996) and Campbell and Levin (2006) for related results. Our main result then shows that the revenue can be further increased by distorting the allocation even further away from the high signal bidders than achieved by posted prices, while maintaining other features of the posted prices, such as constant transfers and constant (interim) allocation probabilities.

As we alluded to above, another interpretation of the pure common value model is that the bidders have independent private values, but that the allocation of the good will be followed by a frictionless resale market, in which values become complete information and the interim owner of the good can make a take-it-or-leave-it offer to the bidder with the highest value. Thus, whoever wins the good in the first stage will earn revenue from resale equal to the highest of the bidders' private values. Now, as the allocation rule of the optimal mechanism favors lower signals, in effect, it induces a more active resale market. The reason is that only the bidder with the highest signal has private information that is payoff relevant, so that discriminating against this bidder reduces the total amount of information rents that bidders receive. Admittedly, this model of resale abstracts from the bidders' incentives to signal their values through the outcome of the auction, and instead emphasizes the common

[3]This version of the common value was studied by Myerson (1981). Bulow and Klemperer (2002) refer to it as the "wallet game," with the interpretation that each bidder privately observes the amount of money in his or her wallet, and bidders are bidding for the amount of money in all the wallets.

value structure that arises from the bidders' ability to resell the good in the same market. A similar model has been studied by Gupta and LeBrun (1999) and Haile (2003) under the assumption that the mechanism used to initially allocate the good is a first-price auction. By contrast, we treat the mechanism as an endogenous object, and derive the auction format that a revenue maximizing seller would use. A similar perspective is taken in recent work by Carroll and Segal (2016) who design the optimal auction in the presence of a resale market. They derive the optimal auction as a maxmin problem where nature chooses the resale market, in terms of information disclosure and bargaining power that is least favorable to the revenue maximizing seller. Their solution and their argument are very different from the ones presented here. In particular, they establish that the least favorable resale market is then one where the bidder with the highest value – independent of his ownership – has the bargaining power and complete information. Thus, he can make a take it or leave it offer to the current owner of the object. By contrast, our resale interpretation implicitly requires that it is the current owner of the object – independent of his value – who has the bargaining power and has complete information.

As a side benefit of our analysis, we show that there is a remarkable connection between incentive compatible allocations in the maxim common value model and those that are incentive compatible in the independent private value setting. We can associate each maximum common value model model with an independent private value model that has the same distribution of signals. Under the former, the value is common and equal to the maximum of the signals, and in the latter, each bidder's signal is equal to his private value. For a given mechanism, the equilibrium strategies could be quite different under the two models. However, if a mechanism implements an allocation in the IPV setting that is conditionally efficient (i.e., conditional on the good being allocated, it is allocated to the bidders' with the highest values), then the same strategies would also be an equilibrium under the analogous maximum common value model. Thus, even though values are uniformly higher under the maximum common value interpretation of the signals, the two models are in a sense "strategically equivalent" as long as the mechanism discriminates in favor of types with higher signals. This result generalizes an observation of Bulow and Klemperer (2002) that bidding one's signal is an equilibrium of the second-price auction in the maximum common value model, as it would be with independent private values. In Bergemann, Brooks, and Morris (2016), we show that the model of the maximum of independent signals attains the minimum revenue for a first-price auction, across all type spaces with a fixed marginal distribution over a pure common value. In combination with the strategic equivalence result, we conclude that first-price auctions generate greater worst-case revenue across all type spaces and equilibria than does any mechanism that implements conditionally efficient allocations in the corresponding IPV setting.

The rest of this paper is organized as follows. Section 2 describes our model. Section 3 generalizes the revenue equivalence formula to the maximum common value model we consider. Section 4 solves for the optimal mechanism. Section 5 concludes with a discussion of properties of the optimal mechanism, and also draws additional connections to the auction theory literature.

2 Model

There are N potential bidders of a single unit of a good, indexed by $i \in \mathcal{N} = \{1, \ldots, N\}$. Each bidder receives a signal $s_i \in S = [\underline{s}, \overline{s}]$ about the good's value. The signals s_i are independent draws across the bidders from an absolutely continuous cumulative distribution $F(s_i)$ with density $f(s_i)$. The bidders all assign the same value to the good, which is the maximum of the signals:

$$v(s_1, \ldots, s_N) = \max\{s_1, \ldots, s_N\}. \tag{1}$$

The common value of the object is thus the maximum of N independent signals. The distribution of signals, $F(s_i)$, induces a distribution $G(v)$ over the common value:

$$G(v) = (F(s))^N.$$

Alternatively, we can let the signals describe a specific *common value model*. With this interpretation, we can take the prior distribution $G(v)$ of the pure common value as given and then type space is chosen such that the maximum of the independently distributed types s_i is equal to the pure common value. We often simply refer to the maximum common value model when we talk about the "maximum of independent signals" common values model.

The bidders are expected utility maximizers, with quasilinear preferences over the good and transfers t_i. Thus, the ordering over pairs (q, t) of probabilities of receiving the good and net transfers to the seller is represented by the utility index:

$$u(s, q, t) = v(s)\, q - t.$$

The good is sold via an auction. Informally, an auction consists of sets of messages and functions that assign to each bidder i a probability of receiving the good and a transfer to the seller. Following Myerson (1981), we invoke the revelation principle and restrict attention to *direct mechanisms*, whereby each bidder simply reports his own signal, and the set of possible message profiles is S^N. The probability that bidder i receives the good given signals $s \in S^N$ is $q_i(s) \geq 0$, with $\sum_{i=1}^N q_i(s) \leq 1$. The *interim* probability that bidder i receives the good is denoted by:

$$Q_i(s_i) = \int_{s_{-i} \in S^{N-1}} q_i(s_i, s_{-i}) f_{-i}(s_{-i}) \, ds_{-i}, \tag{2}$$

where $f_{-i}(s_{-i}) = \prod_{j \neq i} f(s_j)$ is the distribution of signals for bidders $j \neq i$.

The transfer of bidder i to the seller is $t_i(s)$ and the *interim* expected transfer is denoted by:

$$T_i(s_i) = \int_{s_{-i} \in S^{N-1}} t_i(s_i, s_{-i}) f_{-i}(s_{-i}) \, ds_{-i}, \tag{3}$$

The revenue from the direct mechanism is simply the expected sum of transfers:

$$R = \sum_{i=1}^{N} \int_{s_i \in S} T_i(s_i) f(s_i) \, ds_i,$$

and bidder i's surplus from reporting a signal s_i' when his true signal is s_i is

$$U_i(s_i, s_i') = \int_{s_{-i} \in S^{N-1}} q_i(s_i', s_{-i}) v(s_i, s_{-i}) f_{-i}(s_{-i}) \, ds_{-i} - T_i(s_i').$$

We let $U_i(s_i) = U_i(s_i, s_i)$ for short.

We say that the direct mechanism $\{q_i, t_i\}_{i=1}^N$ is *incentive compatible* if

$$U_i(s_i) \geq U_i(s_i, s_i'),$$

for all i and $s_i, s_i' \in S$. The mechanism is *individually rational* if

$$U_i(s_i) \geq 0,$$

for all i and $s_i \in S$. The seller's problem is to maximize R over all incentive compatible and individually rational direct mechanisms $\{q_i, t_i\}_{i=1}^N$.

3 A Revenue Equivalence Formula

A standard tool in optimal auction design is the revenue equivalence formula (Myerson, 1981). In this section, we extend the standard revenue equivalence result to the present setting. For this, it will be useful to distinguish between the winning probability of bidder i

when i himself has the highest signal realization x, and when somebody including i has the highest signal realization x. Let

$$\widehat{Q}_{i,j}(x) = \int_{s_{-j} \in [\underline{s}, x]^{N-1}} q_i(x, s_{-j}) f_{-j}(s_{-j}) \, ds_{-j}, \tag{4}$$

and thus, $\widehat{Q}_{i,j}(x)$ is the likelihood conditional on bidder j's signal being x, that (i) the highest signal is x and (ii) bidder i is allocated the good.[4] We will represent the indirect utility function of bidder i as a function of the probability

$$\widehat{Q}_i(x) \equiv \widehat{Q}_{i,i}(x), \tag{5}$$

that is the probability that bidder i is allocated the good *and* that bidder i has the high signal, conditional on i's signal being x.

By contrast, we denote the total probability that bidder i is allocated the good and that the highest signal is x by:

$$\overline{Q}_i(x) \equiv \sum_{j=1}^{N} \widehat{Q}_{i,j}(x). \tag{6}$$

We should emphasize that these probabilities, $\widehat{Q}_i(x)$ and $\overline{Q}_i(x)$, both differ from the interim probability of winning, $Q_i(s_i)$ defined earlier in (2), in that they represent the probability of the event that bidder i is allocated the good *and* that the highest signal is x, conditional on some bidder having the signal of x. In the case of $\widehat{Q}_i$, it is bidder i who has the signal x, and with $\overline{Q}_i$, it could be any one of the N bidders. Finally, we denote the aggregate probability of allocating the good, respectively by:

$$\widehat{Q}(x) = \sum_{i=1}^{N} \widehat{Q}_i(x) \text{ and } \overline{Q}(x) = \sum_{i=1}^{N} \overline{Q}_i(x). \tag{7}$$

Proposition 1 (Envelope Formula)**.**
In any incentive compatible mechanism, the indirect utility function must satisfy

$$U_i(s_i) = U_i(\underline{s}) + \int_{x=\underline{s}}^{s_i} \widehat{Q}_i(x) \, dx. \tag{8}$$

[4] The objects $\widehat{Q}_{i,j}$, $\widehat{Q}_i$, and $\overline{Q}_i$ (which will be described shortly) are defined so as to maintain as tight a correspondence as possible between the statements of our results and those of Myerson (1981). In particular, $\widehat{Q}_i(x)$ plays the same role in our envelope characterization of bidder surplus as does $Q(x)$ in Myerson's Lemma 2, as defined in his equation (4.1).

Note that in the standard (private value) revenue equivalence formula, the derivative of the indirect utility is the total probability that bidder i is allocated the good conditional on his own signal. In contrast, Proposition 1 says that the derivative of the indirect utility is the probability $\widehat{Q}_i(x)$ that bidder i is allocated the good conditional on his own signal being the highest. The reason is that each bidder only receives information rents from local deviations when the valuation $v(s)$ is sensitive to his private information s_i. In the present pure common value environment, this only occurs when the bidder in question has the highest signal. The above formula is nearly identical to that given by Bulow and Klemperer (1996), with the minor exception that for their derivation they require that the value function $v(s_1,\ldots,s_N)$ is differentiable and strictly increasing, neither of which is the case in the current setting.

Proof of Proposition 1. The proof follows closely that of Lemma 2 in Myerson (1981). Let

$$X([a,b]) = \left\{ s_{-i} \in S^{N-1} \,\middle|\, \max_{j\neq i} s_j \in [a,b] \right\}.$$

If $s_i \leq s_i'$, then

$$\begin{aligned}
U_i(s_i', s_i) &= s_i'\widehat{Q}_i(s_i) + \int_{s_{-i}\in X([s_i,\bar{s}])} \left(\max_j s_j\right) q_i(s_i, s_{-i}) f_{-i}(s_{-i})\, ds_{-i} - T_i(s_i) \\
&\geq s_i'\widehat{Q}_i(s_i) + \int_{s_{-i}\in X([s_i,\bar{s}])} \left(\max_{j\neq i} s_j\right) q_i(s_i, s_{-i}) f_{-i}(s_{-i})\, ds_{-i} - T_i(s_i).
\end{aligned}$$

Thus,

$$U_i(s_i', s_i) - U_i(s_i) \geq (s_i' - s_i)\,\widehat{Q}_i(s_i),$$

and hence

$$U_i(s_i') \geq U_i(s_i) + (s_i' - s_i)\,\widehat{Q}_i(s_i).$$

Similarly,

$$\begin{aligned}
U_i(s_i, s_i') &= s_i\widehat{Q}_i(s_i') + \int_{s_{-i}\in X([s_i,s'])} \left(\max_{j\neq i} s_j - s_i\right) q_i(s_i', s_{-i}) f_{-i}(s_{-i})\, ds_{-i} \\
&\quad + \int_{s_{-i}\in X\left(\left[s_i',\bar{s}\right]\right)} \left(\max_{j\neq i} s_j\right) q_i(s_i', s_{-i}) f_{-i}(s_{-i})\, ds_{-i} - T_i(s_i') \\
&\geq s_i\widehat{Q}_i(s_i') + \int_{s_{-i}\in X\left(\left[s_i',\bar{s}\right]\right)} \left(\max_j s_j\right) q_i(s_i', s_{-i}) f_{-i}(s_{-i})\, ds_{-i} - T_i(s_i')
\end{aligned}$$

Thus,

$$U_i(s_i, s_i') - U_i(s_i') \geq (s_i - s_i')\,\widehat{Q}_i(s_i'),$$

and hence

$$U_i(s_i) + (s_i' - s_i)\widehat{Q}_i(s_i') \geq U_i(s_i').$$

We conclude that

$$U_i(s_i + \Delta) - U_i(s_i) \geq \Delta\widehat{Q}_i(s_i),$$

and hence

$$U_i(s_i) - U_i(s_i - \Delta) \leq \Delta\widehat{Q}_i(s_i),$$

so that $U_i(s_i)$ is differentiable and $U_i'(s_i) = \widehat{Q}_i(s_i)$. □

We can express the total surplus realized in the auction using the total probability $\overline{Q}(x)$as:

$$TS = \int_{x \in S} x\overline{Q}(x)\, g(x)dx.$$

We can then express the revenue of any direct mechanism in terms of $\overline{Q}$ and $\widehat{Q}$:

Proposition 2 (Revenue Equivalence)**.**
The expected revenue from the direct mechanism $\{q_i, t_i\}_{i=1}^N$ *is*

$$R = \int_{x \in S} \left(x\overline{Q}(x) - \int_{y=\underline{s}}^{x} \widehat{Q}(y)\, dy \right) f(x)\, dx - \sum_{i=1}^{N} U_i(\underline{s}). \tag{9}$$

As a result, if two mechanisms induce the same allocation and assign the same utilities to the lowest type $\underline{s}$*, then they must generate the same expected revenue.*

Proof. This follows from Proposition 1 and the formula from total surplus, since

$$R = TS - \sum_{i=1}^{N} \int_{x=\underline{s}}^{\overline{s}} U_i(x)\, f(x)\, dx$$

and

$$\sum_{i=1}^{N} U_i(x) = \int_{y=\underline{s}}^{x} \widehat{Q}(y)\, dy + \sum_{i=1}^{N} U_i(\underline{s}).$$

□

In other words, revenue is simply the total surplus generated by the allocation less the bidders' total information rents. These quantities can be calculated by integrating over the highest signal—i.e., the value.

4 Optimal Revenue

We now characterize the revenue maximizing mechanism. In the classical analysis of Myerson (1981), a regularity condition guarantees that the optimal mechanism is completely characterized by the local incentive constraints as expressed in the envelope characterization of transfers. Under the regularity condition, a revenue formula analogous to (9) has a pointwise maximum. The implied allocation turns out to satisfy a form of monotonicity that is sufficient to guarantee global incentive compatibility. In our environment, however, such local constraints are never sufficient. The revenue equivalence formula tells us that a bidder receives information rents only when he is allocated the good *and* when has the highest signal. If we were only concerned with maximizing the revenue formula from Proposition 2, then the seller could specify an allocation q_i in which the good is always sold to one of the bidders whose signal is less than the maximum. According to the equation (9), bidders would not receive any information rents, and the seller would extract the full surplus as revenue.

This mechanism would, however, violate global incentive constraints, for the simple reason that the bidders would want to misreport lower signals. For example, the bidder with the highest signal $\overline{s}$ would never be allocated the good under this mechanism, and by assumption receives zero rents (i.e., there is no subsidy from the seller), while the bidder with the lowest signal $\underline{s}$ is allocated the good with probability $1/(N-1)$ and pays a $1/(N-1)$ share of the expectation of the highest of the $N-1$ other signals:

$$\hat{s} = \int_{x=\underline{s}}^{\overline{s}} x\,(N-1)\,F^{N-2}(x)\,f(x)\,dx.$$

Thus, the highest type could pretend to be the lowest type and obtain $(\overline{s}-\hat{s})/(N-1)$ for sure. We conclude that in order to characterize the optimal mechanism, we will have to explicitly incorporate global constraints into the optimization problem.

In principle, we might have to consider all of the global deviations whereby a type s_i misreports some $s_i' \neq s_i$. It turns out, however, that maximum revenue in the maximum model is pinned down by a relatively small one-dimensional family of constraints of the following form: instead of reporting signal s_i, report a random signal s_i' that is drawn from the truncated prior $F(s_i')/F(s_i)$ on the support $[\underline{s}, s_i]$. We will refer to this deviation as *misreporting a redrawn lower signal.* Obviously, for a direct mechanism to be incentive compatible, bidders must not want to misreport in this manner. We will presently use these incentive constraints to derive an upper bound on maximum revenue. As part of the derivation, we will also identify features that an allocation would have to satisfy in order to attain the bound, which we will use in the next section to construct a revenue maximizing mechanism.

4.1 An Upper Bound on Revenue

Let us proceed by explicitly describing the incentive constraint associated with misreporting a redrawn lower signal. As it is always possible to increase revenue by reducing the information rent of the lowest signal $U_i(\underline{s})$, we assume throughout the rest of this section that $U_i(\underline{s}) = 0$ for all i. Consequently, the equilibrium surplus of a bidder with type x is

$$U_i(x) = \int_{y=\underline{s}}^{x} \widehat{Q}_i(y)\,dy.$$

In addition, the surplus from misreporting the redrawn lower signal must be

$$\frac{1}{F(x)} \int_{y=\underline{s}}^{x} U_i(x,y)\,f(y)\,dy = \frac{1}{F(x)} \int_{y=\underline{s}}^{x} \left[(x-y)\,\overline{Q}_i(y)\,dy + \int_{z=\underline{s}}^{y} \widehat{Q}_i(z)\,dz\right] f(y)\,dy. \quad (10)$$

This formula deserves a bit more explanation. The second piece inside the brackets is simply the rent that type y receives in equilibrium, which depends on the allocation when y is the highest signal. Of course, this is not the surplus that the deviator would obtain, since in cases where $y < x$, x is the highest signal rather than y. What is the additional surplus that the downward deviator must obtain? While the gains may vary depending on the realized misreport, the average gains across all misreports is relatively easy to compute. Recall that $\overline{Q}_i(y)$ is the total probability that bidder i is allocated the good and that y is the highest type, conditional on some representative bidder having a signal y. The probability that bidder i is allocated the good when the highest signal is y may depend on the particular misreport, but since the misreport is redrawn from the prior, it must be that bidder i is equally likely to fall anywhere in the distribution of signals, so that unconditional on the misreport, $\overline{Q}_i(y)\,f(y)$ is precisely the ex-ante likelihood that i receives the good and y is the highest among the N reported signals. Moreover, if that highest report is less than x, then the surplus that bidder i obtains from being allocated the good is x rather than y, so that $x-y$ is the difference between the deviator's surplus and the equilibrium surplus.

Thus, misreporting a redrawn lower signal is not attractive if and only if

$$\int_{y=\underline{s}}^{x} (x-y)\,\overline{Q}_i(y)\,f(y)\,dy \leq \int_{y=\underline{s}}^{x} \widehat{Q}_i(y)\,F(y)\,dy$$

for every $x \in S$, where the left-hand side is obtained by integrating (10) by parts and canceling terms. Summing across i, we conclude that the direct mechanism deters misreporting redrawn lower signals only if

$$\int_{y=\underline{s}}^{x} (x-y)\,\overline{Q}(y)\,f(y)\,dy \leq \int_{y=\underline{s}}^{x} \widehat{Q}(y)\,F(y)\,dy. \tag{11}$$

Since $\overline{Q}$ and $\widehat{Q}$ are also sufficient for computing revenue, we can now derive an upper bound on revenue by maximizing (9) over all functions $\overline{Q}$ and $\widehat{Q}$ that satisfy (11) and also satisfy the feasibility constraints

$$0 \leq \overline{Q}(x) \leq NF^{N-1}(x) \text{ and } 0 \leq \widehat{Q}(x) \leq NF^{N-1}(x).$$

These range constraints correspond to the fact that x cannot be the highest signal with probability greater than $N(F(x))^{N-1}$. (Strictly speaking, $\widehat{Q}(x)$ cannot be larger than $\overline{Q}(x)$, though we shall see that this constraint is not binding.)

Note that the expression for revenue given by (9) can be integrated by parts to obtain the equivalent expression

$$\int_{x\in S} \left(x\,\overline{Q}(x)\,f(x) - \frac{1-F(x)}{F(x)}\widehat{Q}(x)\,F(x) \right) dx,$$

and integrating the second term by parts again, we obtain:

$$\int_{x\in S} \left(x\,\overline{Q}(x)\,f(x) - \frac{f(x)}{F^2(x)} \int_{y=\underline{s}}^{x} \widehat{Q}(y)\,F(y)\,dy \right) dx.$$

From this formula, it is clear that revenue is increased by making $\int_{y=\underline{s}}^{x} \widehat{Q}(y)\,F(y)\,dy$ as small as possible, so that the constraint (11) must bind everywhere at an optimum. As a result, we can solve out $\widehat{Q}$ in terms of $\overline{Q}$:

$$\widehat{Q}(x) = \frac{1}{F(x)} \int_{y=\underline{s}}^{x} \overline{Q}(y)\,f(y)\,dy,$$

and then substitute this in to obtain the following expression for revenue:

$$\int_{x\in S} \left(x\overline{Q}(x)\,f(x) - \frac{1-F(x)}{F(x)} \int_{y=\underline{s}}^{x} \overline{Q}(y)\,f(y)\,dy \right) dx.$$

Integrating by parts one last time, we obtain our final formula for revenue, which is

$$R = \int_{x \in S} \psi(x) \overline{Q}(x) f(x) \, dx \tag{12}$$

where

$$\psi(x) = x - \int_{y=x}^{\overline{s}} \frac{1 - F(y)}{F(y)} dy,$$

which is strictly increasing and finite valued for $x > \underline{s}$ and positive for x sufficiently close to $\overline{s}$, though it is possible that $\lim_{x \to \underline{s}} \psi(x) = -\infty$. In a sense, $\psi(x)$ is the correct analog of the "virtual value" from Myerson (1981), in that it describes the seller's marginal revenue of allocating the good when the value is x, which is the value itself less the information rents that must be given to deter local deviations and to deter bidders from misreporting redrawn lower signals.

Let $r = \inf \{x | \psi(x) > 0\}$, which must exist and be strictly positive. It is now clear that the pointwise optimum of the revenue formula (12) is given by:

$$\overline{Q}(x) = \begin{cases} 0 & \text{if } x < r; \\ NF^{N-1}(x) & \text{otherwise.} \end{cases} \tag{13}$$

We thus have proved the following:

Proposition 3 (Upper Bound on Revenue)**.**
The revenue of the optimal auction is bounded above by

$$\overline{R} = \int_{x=r}^{\overline{s}} \psi(x) NF^{N-1}(x) f(x) \, dx. \tag{14}$$

In sum, the bound is generated by an allocation that favors low-signal bidders as much as possible by making $\widehat{Q}$ as small as possible. Under the resale interpretation, this means that the seller wants to bias the allocation towards those bidders who are likely to want to resell the good in the secondary market, since they have less private information about the resale value that the seller would have to incentivize them to reveal. $\widehat{Q}$ cannot be too low, however, or else bidders would want to deviate by misreporting redrawn lower signals. This constraint boils down to the requirement that $\widehat{Q}(x)$ cannot be smaller than the probability that the good is allocated conditional on the highest signal being less than x.

Thus, increasing $\overline{Q}(x)$ has two competing effects on revenue: it increases the total surplus that is generated by the auction, but it also generates additional information rents for types that are greater than x since it increases the value of misreporting a redrawn lower signal.

The function $\psi(x)$ represents the net contribution to revenue of allocating the good when one takes into account both of these forces, and the allocation that maximizes revenue is bang-bang: allocate the good if and only if $\psi(x) \geq 0$.

4.2 An Optimal Mechanism

We now construct a direct mechanism that attains the bound described in Proposition 3. Let

$$\gamma(x) = \frac{1}{N}\left(1 - \left(\frac{F(r)}{F(x)}\right)^N\right). \tag{15}$$

The allocation is as follows: if the highest signal x is at least r, then the good is allocated to the bidder with the highest signal with probability $\gamma(x)$, and with probability $1-\gamma(x)$ the good is allocated to one of the $N-1$ other bidders who do not have high signals at random. If the highest signal is less than r, then the good is not allocated all. Formally, the probability by which bidder i receives the object when the realized profile of signal is s is given by:

$$q_i(s) = \begin{cases} \gamma(\max s), & \text{if } s_i > s_j \ \forall j \neq i \text{ and } s_i \geq r; \\ \frac{1}{N-1}(1-\gamma(\max s)), & \text{if } s_i < \max s \text{ and } \max s \geq r; \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

(We have ignored ties, which occur with probability zero.)

This mechanism is reverse engineered to implement the allocation corresponding to the solution to the relaxed program. Observe that the good is always allocated as long as the highest signal is at least r. Thus, total surplus under this mechanism would coincide with that attained in the solution to the relaxed program. In addition, note that

$$\gamma(x) = \frac{\widehat{Q}(x)}{NF^{N-1}(x)}$$

where

$$\widehat{Q}(x) = \frac{1}{F(x)}\int_{y=\underline{s}}^{s} \overline{Q}(y) f(y)\, dy = \frac{F^N(x) - F^N(r)}{F(x)}$$

is the optimal value of $\widehat{Q}$ for the relaxed program. Recall that $\widehat{Q}$ is the probability, conditional on some bidder having a signal of x, that x is the highest signal and that the high-signal bidder is allocated the good. In a symmetric allocation, each bidder would be equally likely to obtain this allocation, so that the probability that a representative bidder with signal x is allocated the good and has the high signal is $\widehat{Q}(x)/N$. Since $F^{N-1}(x)$ is the probability that a bidder with signal x has the highest signal, the likelihood that bidder i is allocated the good when he has signal x and *conditional* on having the highest signal is exactly $\gamma(x)$. As such, if this mechanism is incentive compatible, it must implement the allocation that maximizes revenue for the relaxed program.

The implied interim transfer is constant in s_i and is equal to

$$T_i(s_i) = T = \int_{x=r}^{\overline{s}} x(1-\gamma(x)) F^{N-2}(x) f(x) dx, \tag{17}$$

which is simply the expected surplus generated by allocating the good to any type $s_i < r$. The transfer can therefore be viewed as an entry fee that is paid independent of the type and the outcome of the auction.

Theorem 1 (Optimal Auction)**.**
The direct mechanism described by (15) - (17) is individually rational and incentive compatible and attains maximum revenue. The transfer payment $T_i(s_i)$ and the probability $Q_i(s_i)$ of receiving the good are constant in s_i.

Proof. We show that the mechanism defined by (15)–(17) is incentive compatible. Consider a type $s_i \geq r$ that misreports to some signal x with $r \leq x < s_i$. The resulting surplus consists of three pieces:

$$U(s_i, x) = s_i \gamma(x) F^{N-1}(x) + \int_{y=x}^{s_i} s_i (1-\gamma(y)) F^{N-2}(y) f(y) dy$$
$$+ \int_{y=s_i}^{\overline{s}} y(1-\gamma(y)) F^{N-2}(y) f(y) dy - T.$$

Differentiating this expression with respect to x, we obtain

$$U'(s_i, x) = s_i \left(\gamma'(x) F^{N-1}(x) + (N\gamma(x) - 1) F^{N-2}(x) f(x)\right).$$

But substituting in the definition of γ, this becomes

$$s_i \left(\frac{F^N(r) f(x)}{F^{N+1}(x)} F^{N-1}(x) + \left(\left(1 - \frac{F^N(r)}{F^N(x)}\right) - 1\right) F^{N-2}(x) f(x) \right) = 0,$$

so that downward deviations are not attractive.

On the other hand, if type $s_i \geq r$ misreports $x > s_i$, then surplus is

$$\begin{aligned} U(s_i, x) &= s_i \gamma(x) F^{N-1}(s_i) + \int_{y=s_i}^{x} y\gamma(x)(N-1) F^{N-2}(y) f(y)\, dy \\ &\quad + \int_{y=x}^{\bar{s}} y(1-\gamma(y)) F^{N-2}(y) f(y)\, dy - T. \end{aligned}$$

Differentiating with respect to x, we now obtain

$$\begin{aligned} U'(s_i, x) &= \gamma'(x)\left(s_i F^{N-1}(s_i) + \int_{y=s_i}^{x} y(N-1) F^{N-2}(y) f(y)\, dy\right) \\ &\quad + (N\gamma(x)-1) x F^{N-2}(x) f(x) \\ &\leq \gamma'(x) x F^{N-1}(x) + (N\gamma(x)-1) x F^{N-2}(x) f(x) = 0 \end{aligned}$$

since

$$s_i F^{N-1}(s_i) + \int_{y=s_i}^{x} y(N-1) F^{N-2}(y) f(y)\, dy \leq x F^{N-1}(x).$$

As a result, upward deviations are not attractive either.

Using (15) we can compute the interim probability of winning for $s_i \geq r$:

$$\begin{aligned} Q_i(s_i) &= F(s_i)^{N-1}\left(\frac{1}{N}\left(1-\left(\frac{F(r)}{F(s_i)}\right)^N\right)\right) \\ &\quad + \left(1-F(s_i)^{N-1}\right)\int_{y=s_i}^{\bar{s}} \frac{\left(\frac{1}{N}+\frac{1}{N(N-1)}\left(\frac{F(r)}{F(y)}\right)^N\right)(N-1) f(y) F(y)^{N-2}}{1-F(s_i)^{N-1}} dy \\ &= \frac{1}{N}\left(1-\frac{F(r)^N}{F(s_i)}\right) + \int_{y=s_i}^{\bar{s}} \frac{1}{N}\frac{F(r)}{F(y)^2}^N f(y) dy \\ &= \frac{1}{N}(1-F(r)^N). \end{aligned}$$

Similarly, if $s_i < r$, then the first term above drops out, and we have

$$\begin{aligned} Q_i(s_i) &= \left(1-F(r)^{N-1}\right)\int_{y=r}^{\bar{s}} \frac{\left(\frac{1}{N}+\frac{1}{N(N-1)}\left(\frac{F(r)}{F(y)}\right)^N\right)(N-1) f(y) F(y)^{N-2}}{1-F(r)^{N-1}} dy \\ &= \frac{1}{N}\left(1-F(r)^{N-1}\right) + \int_{r}^{\bar{s}} \frac{1}{N}\frac{F(r)}{F(y)^2}^N f(y) dy \\ &= \frac{1}{N}(1-F(r)^N), \end{aligned}$$

which concludes the proof of our main result. $\square$

The optimal mechanism thus offers a constant interim payment and probability of winning the good. But there is an important difference to a posted price mechanism. The probability of receiving the good given a type profile s is not uniformly distributed, but rather biased away from the bidder with the highest signal. Nonetheless, the interim probability of receiving the good must be constant across the types. To see this, consider the highest type of a bidder. If he is indifferent to *randomly* redrawing a lower signal, then in equilibrium he must also be indifferent to any *pointwise* downward deviation. Now, this type knows that the value is exactly the highest signal, so that all he cares about is the total expected probability of getting the good and the expected transfer. Since the latter is constant, the former must be as well. Moreover, since the signals are independent, this expected probability of getting the good can only depend on the message, not the type that is sending the message.

But since the other types separate, it must be that the correlation between getting the good and others signals depends on the report. Specifically, it must be that a lower report means you are more likely to get the good when others have higher signals. Thus, a high type is happy to deviate down (since he doesn't care what others types are, conditional on being less than his) but a low type would not want to deviate up, because then he would win more often when others signals are relatively low (but still higher than his, so that they mean the value is lower), which is less desirable.

We illustrate the nature of the optimal auction with the uniform distribution of values, $G(v) = v$. The corresponding distribution of signals by the bidders is given by $F(x) = x^{1/N}$. In this case, the generalized virtual utility $\psi(x)$ takes the form:

$$\psi(x) = x - \int_{y=x}^{1} \left(x^{-\frac{1}{N}} - 1\right) dx = \frac{1}{N-1}\left(Nx^{\frac{N-1}{N}} - 1\right).$$

The optimal cutoff r is therefore

$$r = \left(\frac{1}{N}\right)^{\frac{N}{N-1}}, \tag{18}$$

which is strictly decreasing in N. Optimal revenue can be computed using $F(x) = x^{1/N}$ and the optimal cut-off above as:

$$\frac{1}{N-1}\int_{x=r}^{1}\left(Nx^{\frac{N-1}{N}}-1\right)dx = \frac{1}{N-1}\left[\frac{N^2}{2N-1}x^{\frac{2N-1}{N}}-x\right]_{x=r}^{1}$$
$$= \frac{1}{2N-1}\left(N-1-\frac{1}{N}^{\frac{N}{N-1}}\right).$$

The revenue is strictly increasing in N as well and converges against the expected value of the object equal to $1/2$ and hence full surplus extraction as N grows large. We will return to this example in our discussion below.

5 Discussion

We begin this section by suggesting an indirect implementation of the optimal auction by a descending clock auction. We then relate the equilibrium behavior in the current common value environment to the equilibrium behavior independent private value environment. We then revisit the classic result of Bulow and Klemperer (1996) that compares the revenue of the optimal auction with N bidders with the second price auction without reserve price with $N+1$ bidders. Finally, we return to the resale interpretation of our model.

5.1 Descending Clock Auction and Posted Prices

While we have described the optimal mechanism in terms of its direct implementation, there is a natural indirect implementation of the optimal mechanism that uses a "descending clock," in a manner that is in a sense dual to a Dutch auction. In the Dutch auction, the value of the clock represents the *price* at which the bidder who stops the clock will purchase the good. In our indirect mechanism, the value of the clock represents the *probability* with which the bidder who stops the clock gets allocated the good.

We can describe this auction more explicitly as follows. First, all of the bidders must pay an entry fee of T to enter the auction, as determined above by (17). Once all of the bidders have entered, there are no subsequent transfers, and the allocation is determined as follows. There is a probability p which starts at $\gamma(\bar{s}) \leq 1/N$ and descends gradually. Similar to the Dutch clock auction described in Milgrom and Weber (1982), the bidders each have a button which is initially depressed, and the auction ends as soon as the first bidder releases his button. If bidder i is the first to release his button at $p > 0$, then bidder i is allocated the good with probability p, and each of the other bidders is allocated the good with probability $(1-p)/(N-1)$. Finally, if p reaches zero, the auction ends and no bidder is allocated the good.

It is not hard to see that there is an equilibrium of this descending clock auction in which each bidder uses the cutoff strategy of staying active until $p \leq \gamma(s_i)$. The reason is that the information which each bidder receives as p descends but as long as $p \geq \gamma(s_i)$ does not change the marginal benefits of staying in versus dropping out, since the derivative of the indirect utility derived above only depends on outcomes when the highest signal is less than $\hat{s}$, where $p(\hat{s})$ is the cutoff that the bidder deviates to. Put differently, suppose we replaced $F(s)$ by a truncated distribution $\widehat{F}(s) = F(s)/F(x)$, which conditions on the knowledge that all signals are less than a bidder's own signal x. Then the form for γ remains exactly the same and the incentive compatibility of truthtelling would continue to hold, thus verifying that bidders still would not want to deviate even after they see that other bidders have not yet ended the auction. Interestingly, even as p gets arbitrarily close to zero, bidders are still willing to wait and see if someone else stops the auction, with the reason being that the probability of being allocated the good as the bidder who stops the auction is sufficiently small compared to the corresponding probability when someone else stops the auction. Thus, bidders with low signals are willing to wait and hope that someone else stops the auction before it is too late. We therefore have:

Proposition 4 (Descending Clock Auction).
The optimal auction can be implemented by a descending clock auction.

The optimal selling mechanism of Theorem (1) is achieved with a constant interim transfer $T = T_i(s_i)$ and a constant interim winning probability $Q = Q_i(s_i)$. But in contrast to a posted price mechanism it distorts the ex post allocation $q_i(s)$ as a function of the threshold value r which the highest signal has to exceed before the object is allocated. This suggests that a posted price mechanism becomes optimal if the threshold value r were to coincide with lowest signal in the support of S, that is if $r = \underline{s}$. Interestingly, this also suggests that if a posted price mechanism is an optimal mechanism, then the mechanism will not exclude any type of bidders. Thus, the posted price is chosen so that every type of bidder is willing to buy the object. We refer to the posted price thus as fully inclusive as it does not exclude any bidders at any signal realization.

Proposition 5 (Posted Prices).
A posted price mechanism is optimal if and only if

$$\underline{s} - \int_{\underline{s}}^{\overline{s}} \frac{1-F(x)}{F(x)} dx \geq 0.$$

If a posted price mechanism is optimal, then it is fully inclusive and the price p is:

$$p = T \cdot N = \int_{\underline{s}}^{\overline{s}} x(N-2)F(x)^{N-1}f(x)dx.$$

The posted price p is equal to the expectation of the highest among $N-1$ signal realizations, which is exactly the expectation that a bidder with lowest possible signal $s_i = \underline{s}$ has about the value of the object. We note in passing that the posted price mechanism is indeed the limit of the descending clock auction as r goes to zero, in which case the initial value of the clock $\gamma(\overline{s})$ converges to $1/N$, and in equilibrium, all types stop the clock immediately.

We can illustrate these results with the uniform example we introduced in the previous section. For the exact posted price result, consider the family of translated uniform distributions which are uniform on $[a, a+1]$. The marginal revenue function for these distributions is

$$\psi_a(x) = x - \int_{y=x}^{a+1} \left((x-a)^{-\frac{1}{N}} - 1 \right) dx,$$

so that the lowest marginal revenue is

$$\begin{aligned}\psi_a(a) &= a - \int_{y=a}^{a+1} \left((x-a)^{-\frac{1}{N}} - 1 \right) \\ &= a - \frac{1}{N-1}.\end{aligned}$$

Thus, for $a > 1/(N-1)$, it is optimal to not exclude any bidders, and a posted price is optimal. Moreover, the posted price is such that every bidder, irrespective of his signal realization, declares his interest to receive the object at the price.

We note that while posted prices need not be optimal for fixed N, they will be approximately optimal for N large. More precisely, suppose we hold fixed the distribution of the common value at $P(v)$ and consider the sequence of economies indexed by N in which bidders receive independent signals from $(P(v))^{1/N}$ and the value is the highest signal. Then if the seller sets a posted price of $t = \int_v vP(dv) - \epsilon$ for some $\epsilon > 0$, then as $N \to \infty$, the probability that at least one of the bidders assigns a value of at least t to the good goes to one, so that the seller can approximately extract the whole surplus. The reason is that for N large, when a bidder has a low signal, the expectation of the highest of the others' signals is converging to the unconditional expectation of the value, and the probability that at least one bidder has a low signal is going to one.

We can also illustrate the limit optimality of posted prices in the uniform case (without translated support and $a = 0$). As $N \to \infty$, the optimal cutoff r as derived in (18) goes to zero and the optimal revenue converges to $1/2$. Indeed, it is impossible for revenue to exceed $1/2$, and we could have separately concluded that this must be attained in the limit, since it is obtained by even simpler mechanisms such as posted prices (which correspond to the case where $r = 0$ for finite N). Specifically, the seller could always sell the good at a price equal to the expectation of the highest of $N-1$ draws. Even the zero type would want to buy at this price, so that the good would always be sold, and revenue would be

$$p = \int_{x=0}^{1} x \frac{N-1}{N} x^{\frac{N-1}{1}} x^{-\frac{N-1}{N}} dx = \frac{1}{2} \frac{N-1}{N}.$$

We can therefore further conclude that the optimal revenue converges to the total surplus at the same rate as would revenue from the posted price under which the good is always sold.

5.2 Comparison with Independent Private Value Environment

In the analysis of the "maximum game," Bulow and Klemperer (2002) in Section 9 show that the second-price auction (or equivalently the ascending auction) has an equilibrium in which bidders bid their signals. In this equilibrium, the bidder with the highest signal wins the auction and pays the second-highest signal. To see this, suppose that bidders $j \neq i$ are bidding their signals. The surplus to a bidder with signal s_i from bidding $b < s_i$ (ignoring ties) is

$$\int_{s_{-i} \in X([\underline{s}, b])} \left(s_i - \max_{j \neq i} s_j \right) f_{-i}(s_{-i})\, ds_{-i},$$

which is clearly increasing in b. On the other hand, if $b > s_i$, then the surplus for the bidder i is

$$\int_{s_{-i} \in X([\underline{s}, s_i])} \left(s_i - \max_{j \neq i} s_j \right) f_{-i}(s_{-i})\, ds_{-i} + \int_{s_{-i} \in X([s_i, b])} \left(\max_{j \neq i} s_j - \max_{j \neq i} s_j \right) f_{-i}(s_{-i})\, ds_{-i}$$

which is equal to the surplus from bidding s_i! Thus, it is optimal to bid any amount which is at least your signal, and, in particular, it is optimal to bid your signal. Thus in the equilibrium of the second price auction, each bidder is indifferent between bidding his signal and bidding any *higher* signal. In sharp contrast, in the optimal auction as established in Theorem (1), each bidder is indifferent between reporting his signal and reporting any *lower* signal. Thus, we find that in the second price auction of this pure common value environment, each bidder behaves as if his signal is his true private value rather than a signal and in particular a lower bound on the pure common value.

This observation can be generalized in the following manner. Consider the alternative model in which each bidder's signal is again drawn from F, but instead of the value being the highest of the signals, the value is the bidder's own signal. In other words, this is the independent private value model, where bidder i's value is

$$v_i (s_1, \ldots, s_N) = s_i.$$

Let $H(s) = \{i | s_i = \max_j s_j\}$ denote the set of bidders with high signals. We will say that the direct mechanism $\{q_i, t_i\}$ is *conditionally efficient* if (i) $q_i(s) > 0$ if and only if $s_i \in H(s)$ and (ii) there exists a cutoff r such that the good is allocated whenever $\max_i s_i > r$.

Proposition 6 (Strategic Equivalence)**.**
Suppose a direct mechanism $\{q_i, t_i\}$ is incentive compatible and individually rational for the independent private value model in which $v_i(s) = s_i$ and that the allocation is conditionally efficient. Then $\{q_i, t_i\}$ is also incentive compatible and individually rational for the maximum common value model in which $v_i(s) = \max_j \{s_j\}$.

Proof of Proposition 6. Let

$$Q_i(s_i) = \int_{s_{-i} \in S^{N-1}} q_i(s_i, s_{-i}) f_{-i}(s_{-i}) \, ds_{-i}$$

denote the probability that bidder i is allocated the good. Since the bidder with the lowest signal $\underline{s}$ is allocated the good with zero probability, it must be that $t_i(\underline{s}) \leq 0$. In addition, conditional efficiency implies that $Q_i(s_i) = \widehat{Q}_i(s_i)$, so that

$$U_i(s_i) - U_i(s_i') = \int_{x=s_i}^{s_i'} Q_i(x) \, dx$$

is satisfied by both the indirect utilities when$v_i = s_i$ and when $v_i = \max s$. As a result, the same transfers satisfy local incentive constraints and individual rationality. We then only need to check global incentive constraints. Let $U_i(s_i, s_i')$ denote the utility of a type s_i that reports s_i' when the value is the maximum signal, and let $\widetilde{U}_i(s_i, s_i')$ denote the same when values are private. Then

$$U_i(s_i, s_i') = \int_{s_{-i} \in S^{N-1}} \max_j s_j q_i(s_i', s_{-i}) f_{-i}(s_{-i}) \, ds_{-i} - T_i(s_i')$$

and

$$\widetilde{U}_i(s_i, s_i') = \int_{s_{-i} \in S^{N-1}} s_i q_i(s_i', s_{-i}) f_{-i}(s_{-i}) \, ds_{-i} - T_i(s_i').$$

If $s_i' < s_i$, then since $q_i(s_i', s_{-i}) = 0$ unless $s_i' \geq s_j$ for all $j \neq i$. As a result, $U_i(s_i, s_i') = \widetilde{U}_i(s_i, s_i')$ and $U_i(s_i) \geq U_i(s_i, s_i')$ follows from the fact that $\widetilde{U}_i(s_i) \geq \widetilde{U}_i(s_i, s_i')$.

So consider the case where $s_i' > s_i$. Conditional efficiency implies that

$$Q_i(s_i) = \mathbb{I}_{s_i \geq r} F^{N-1}(s_i).$$

As a result, for $s_i \geq r$,

$$\begin{aligned}
T_i(s_i) &= s_i F^{N-1}(s_i) - \int_{x=r}^{s_i} F^{N-1}(x)\, dx - U_i(\underline{s}) \\
&= \int_{s_{-i} \in X([r, s_i])} \max_{j \neq i} s_j f_{-i}(s_{-i})\, ds_{-i} - U_i(\underline{s}).
\end{aligned}$$

Thus,

$$\begin{aligned}
U_i(s_i, s_i') &= s_i F^{N-1}(s_i') + \int_{s_{-i} \in X\left(\left[s_i, s_i'\right]\right)} \left(\max_{j \neq i} s_j - s_i\right) f_{-i}(s_{-i})\, ds_{-i} - T_i(s_i') \\
&= s_i F^{N-1}(s_i') + \int_{s_{-i} \in X\left(\left[s_i, s_i'\right]\right)} \left(\max_{j \neq i} s_j - s_i\right) f_{-i}(s_{-i})\, ds_{-i} \\
&\quad - \int_{s_{-i} \in X\left(\left[r, s_i'\right]\right)} \max_{j \neq i} s_j f_{-i}(s_{-i})\, ds_{-i} + U_i(\underline{s}) \\
&= s_i F^{N-1}(s_i) - \int_{s_{-i} \in X([r, s_i])} \max_{j \neq i} s_j f_{-i}(s_{-i})\, ds_{-i} + U_i(\underline{s}),
\end{aligned}$$

which is independent of s_i', thus proving that bidders have no incentive to deviate up. $\square$

As a corollary of this result, consider any mechanism that admits an equilibrium in which the allocation is conditionally efficient in the independent private value model, and such that all actions are used in equilibrium. We will refer to such a mechanism as *standard*. Then the same strategies will also be an equilibrium of that mechanism in the corresponding maximum common value model with the same distribution of signals but in which all bidders have a common value equal to the maximum signal. Moreover, all of these mechanisms (which are revenue equivalent in the IPV setting) will also be revenue equivalent when the value is the maximum signal.

This strategic equivalence is somewhat surprising. When we transform the independent private value model into the common value model, the bidders' interim expectations of their values increase substantially, since their value is now the maximum of their own signal and the highest signal of others. In principle, one might think that the higher values would induce the bidders to bid more aggressively. However, this turns out not to be the case.

The reason is that others' bidding strategies are correlated with the value in such a way that all of the potential surplus gains from more aggressive bidding are *exactly* dissipated by higher sales prices. In fact, while bidders would strictly prefer their equilibrium bids over higher bids in the IPV model, they become *indifferent* between their equilibrium bids and all higher bids when the value is the maximum signal. In other words, the winner's curse is exactly strong enough to make any bid beyond the realized signal just as good as bidding the realized signal.

In related work, (Bergemann et al., 2016), we analyze the range of possible revenue outcomes of the first-price auction, where we held fixed the distribution of bidders' values but varied the bidders' information and equilibrium strategies. The main result, Theorem 1, shows that when bidders' values are common, the information structure that minimizes revenue is precisely the one in which the bidders have independent signals, and the value is the maximum signal. Thus, a further corollary of Proposition 6 is that holding fixed the distribution of the common value, the first-price auction generates greater worst-case revenue over all type spaces and equilibria than any mechanism that induces efficient allocations on independent private value models. To wit, by Proposition 6, we can extend the revenue equivalence result from the independent private value model to the maximum common value model for standard auctions. By Theorem 1 of Bergemann et al. (2016), the lowest revenue in a first price auction is achieved in the maximum type space. By Proposition 6, this revenue is also achieved in all other standard auctions, but in any such auction, the worst case type space could possibly be different, and hence yield even lower revenues.

In fact, in Section 6 of Bergemann et al. (2016), we establish that the worst-case revenue even for first-price auctions with a reserve price with a type space similar to the one considered here, but where low signals are pooled up to some threshold (so that bidders see, for example, the maximum of s_i and a cutoff value r). One can extend the revenue equivalence formula and Proposition 6 to show that first-price auctions with reserve prices generate greater worst-case revenue than any other standard mechanism. We state this as a formal result:

Corollary 1 (Revenue Performance of First-Price Auction)**.**
Suppose there is a pure common value v with fixed distribution $P(v)$. Then there exists a reserve price r such that the first-price auction with minimum bid r generates greater minimum revenue than any standard mechanism, where the minimum is taken across all Bayes Nash equilibria and across all common value common prior type spaces where the distribution of the common value is P.

Note that while our proposition only asserts that first-price auctions are weakly better than other standard mechanisms in the maxmin sense, in some cases we know that the ordering is strict. For example, second-price auctions admit "bidding ring" equilibria in which one bidder bids a large amount while the others bid zero. Indeed, we have performed simulations that indicate that all-pay auctions also have strictly worse minimum revenue than first-price auctions. This revenue ranking contrasts sharply with the ranking of mechanism suggested by "linkage principle" of Milgrom and Weber (1982), who find that that the English auctions generate more revenue than the second-price than the first-price auctions when values are affiliated. (Note that the maximum common value model is an affiliated values model.) The different conclusion here stems from the fact that our worst-case criterion varies the information structure and equilibrium while holding the mechanism constant, whereas Milgrom and Weber's comparison holds the information structure constant while comparing mechanisms.

One can therefore interpret Corollary 1 as an explanation as to why first-price auctions are so much more prevalent than second-price auctions, or any other auction format that maximizes revenue in independent private value settings. The reason is that while all of these mechanisms generate the same revenue in the IPV environment, standard mechanisms other than the first-price auction are more susceptible to low revenue in other informational environments and when equilibrium selection is less favorable to the seller.

5.3 Auctions versus Optimal Mechanism

We derived the optimal mechanism in an environment with independent signals and common values. In a seminal paper, Bulow and Klemperer (1996), established the limited power of optimal mechanisms as opposed to standard auction formats. They showed that the revenue of the optimal auction with N bidders is strictly dominated by a standard auction without reserve prices with $N+1$ bidders. The pure common value environment analyzed here is an instance of their more general interdependent value environment with one exception. The virtual utility function—or marginal revenue function in the language of Bulow and Klemperer (1996)—is not monotone due the maximum operator in the common value model. We saw that this aspect of the environment lead to an optimal mechanism with features distinct from the standard first or second price auction. Namely, the optimal mechanism elicits the information from the bidder with the highest signal but minimizes the probability of assigning him the object subject to the incentive constraint. The optimal mechanism thus implements a very different allocation from a standard auction such as a first or second price auction in which the bidder with the highest signal would typically receive the object with probability one. This raises the question whether the revenue comparison suggested by Bulow and Klemperer (1996) still resolves in favor of the standard auction.

In the pure common value environment considered here, the value of the object is the same for all bidders. However, only the bidder with the highest signal can guarantee himself an information rent. Indeed, the *virtual utility* of each bidder, $\pi_i(s_i, s_{-i})$ is constant in s_i and equal to the utility until s_i becomes the largest signal. At this critical point, the virtual utility of bidder i displays a downward jump, and thereafter has the standard expression of the virtual utility:

$$\pi_i(s_i, s_{-i}) = \begin{cases} \max_j\{s_j\}, & \text{if } \; s_i \leq \max\{s_{-i}\}; \\ \max\{s_j\} - \frac{1-F_i(s_i)}{f_i(s_i)}, & \text{if } \; s_i > \max\{s_{-i}\}. \end{cases} \tag{19}$$

The downward discontinuity in the virtual utility indicates why the seller wishes to minimize the probability of assigning the object to the bidder with the high signal. We notice that the downward discontinuity is due to the value function of the bidders, and arises independent of the nature of the distribution function. The virtual utility of bidder i therefore fails the monotonicity assumption even when the hazard rate of the distribution function is increasing everywhere. Bulow and Klemperer (1996) required the monotonicity of the virtual utility when establishing their main result that an absolute English auction with $N+1$ bidders is more profitable than any optimal mechanism with N bidders.

Indeed, we can show that the revenue ranking established in Bulow and Klemperer (1996) does not extend to the current auction environment. For this, it will suffice to restrict ourselves to the class of power distribution functions:

$$G(v) = v^\alpha, \; v \in [0,1], \; \alpha \in \mathbb{R}_+.$$

We have the following result and the proof is relegated to the Appendix.

Proposition 7 (Revenue Comparison)**.**
For every $N \geq 2$, there exists $\bar{\alpha}$, with $1 < \bar{\alpha} < \infty$, such that an optimal auction with N bidders is more profitable than a second price auction with $N+1$ bidders if and only if $\alpha < \bar{\alpha}$.

In the special case of the uniform distribution, thus $\alpha = 1$, the optimal mechanism with N bidders is therefore more profitable than a second price auction with $N+1$ bidders irrespective of the number N of bidders. The analysis of the optimal auction shows that the optimal auction uses a fully inclusive posted price for all values of $\alpha$and N. By using a posted price, the optimal auction avoids low revenue realizations that arise when even the highest realized signal is low. However as α increases, the distribution of values becomes more concentrated around 1, and the probability of low value realizations decreases. Thus eventually the capacity of the optimal auction to avoid low revenue realization is overshadowed by an additional signal realization and hence the possibility of higher revenues with an additional bidder. This explains that eventually as α increases, the revenue ranking established in Bulow and Klemperer (1996) is reestablished.

5.4 Auction with Resale

We conclude by revisiting the resale interpretation of the "maximum independent signal" model. We observed that a leading interpretation of the maximum comm value model comes from re-sale in a model with independent private values. The characterization of the optimal mechanism remains valid if we interpret the model as one where the object is initially sold optimally among N bidders with independent private values, and then possibly offered for resale under complete information. In contrast to previous work on auctions with resale, such as Gupta and LeBrun (1999) and Haile (2003), that also study resale under complete information, our analysis investigates the optimal mechanism in the primary market. The earlier literature started with the assumption that the mechanism used to initially allocate the good is a first-price auction to be followed by an optimal take-it-or-leave-it offer.

The resale interpretation has some limitations. Truth telling is an equilibrium of this mechanism under the assumption that values automatically become complete information in the secondary market, so that the resale price is exactly the highest value. Truthtelling would no longer be incentive compatible if bidders had to infer one another's values from the outcome of the auction. To see this, recall that bidders are indifferent between reporting their true type and reporting any lower type when there is complete information. Intuitively, when a bidder has the highest signal, they only receive rents from being allocated the good outright since otherwise they have to buy it at its value in the secondary market. The optimal mechanism makes it so that the probability of being allocated the good is independent of the report $s' < s_i$, conditional on s_i being the highest signal, so that bidders do not benefit from deviating down. It is essential for this logic that the bidder not make any rents in the resale market, even after they report a lower signal. On the other hand, if by reporting a lower signal a bidder could signal a lower willingness to pay, then the deviator could buy the good in the resale market at a price strictly less than its value, so that the downward deviator would be strictly better off. It remains an open question what would be the form of the revenue maximizing mechanism if resale prices were influenced by the auction format.

6 Conclusion

We have characterized novel revenue maximizing auctions for a class of common value models. These common value models have the qualitative feature that values are more sensitive to the private information of bidders with more optimistic beliefs. This seems like a natural feature of many economic environments, in which the most optimistic bidder has the most useful information for determining the best-use value of the good, and therefore has a greater information rent. One class of models for which this is undoubtedly the case is when the bidders have private values but the auction will be followed by a friction less resale market, so that the total surplus generated by allocating the good is always the highest private value. In contrast, the characterizations of optimal revenue that exist in the literature depend on information rents being smaller for bidders who are more optimistic about the value.

The qualitative impact is that while earlier results found that optimal auctions discriminate in favor of more optimistic bidders, we find that optimal auctions discriminate in favor of less optimistic bidders, since they obtain less information rents from being allocated the good. In certain cases, the optimal auction reduces to a fully inclusive posted price, under which the likelihood that a given bidder wins the good is independent of their private information. In many cases, however, the optimal auction strictly favors bidders whose signals are not the highest. This is necessarily the case when there is no gap between the seller's cost and the support of bidder's values.

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7 Appendix

Proof of Proposition 7. We compare the revenue of an optimal mechanism with N bidders with a second price auction (without reserve prices) with $N+1$ bidders.

We begin with the second price auction with $N+1$ bidders. By Proposition 6, here every bidder bids his signal, and the realized price is the second highest signal. The expected revenue is therefore the expectation of the second order statistic, $s^{(2)}$, which has a density (for N independent and identically distributed random variables):

$$h(s) = N(N-1)(1-F(s))F(s)^{N-2}f(s).$$

We can then evaluate the second order statistic for $N+1$ bidders. Suppose the distribution of values is given by $G(v) = v^{\alpha}$, then the signal distribution of each of $N+1$ bidders is given by $F(s) = s^{\frac{\alpha}{N+1}}$. The resulting expected revenue is then given by

$$\begin{aligned}
&\int_0^1 s(N+1)N(1-F(s))F(s)^{N-1}f(s)ds \\
=&\int_0^1 s(N+1)N\left(1-s^{\frac{\alpha}{N+1}}\right)s^{\frac{\alpha(N-1)}{N+1}}\frac{\alpha}{N+1}s^{\frac{\alpha}{N+1}-1}ds \\
=&\frac{\alpha^2}{\alpha+1}\frac{N}{(N+1)(1+\alpha)-\alpha}.
\end{aligned} \tag{20}$$

We can separately compute the expected revenue from the optimal mechanism with N bidders. In the comparison across the two settings we follow the method of Bulow and Klemperer (1996). Namely, when we restrict attention to N – as opposed to $N+1$ – bidders, the distribution of signals and values does not change. Rather, the mechanism is restricted to N bidders, and neither the mechanism nor the N bidders observe the signal realization of the $N+1$-th bidder. Instead, the seller and the N bidders take the expectation over the (unobserved) signal realization of the $N+1$-th bidder.

Thus, in the optimal mechanism with N bidders, the true value is given by

$$v = \max\{s_1, ..., s_{N+1}\},$$

but every bidder $i < N+1$ with signal realization s_i has to compute the expectation of the true value conditional on s_i being the highest signal among the realized N signals. The conditional distribution is given by

$$G(v\,|s\,) = \begin{cases} 0 & \text{if} \quad v < s; \\ F(v) & \text{if} \quad v \geq s; \end{cases}$$

and therefore the conditional expectation is given by:

$$\mathbb{E}\left[v\,|s\right] = sF(s) + \int_s^1 yf(y)dy > s. \tag{21}$$

In the optimal mechanism with N bidders, we therefore replace the conditional expectation of the true value conditional on s being the highest signal among the realized $N+1$ signals (which is s) by the conditional expectation of the true value conditional on s being the highest signal among the realized N signals (which is $\mathbb{E}\left[v\,|s\right] > s$).

We restrict attention to the class of power distribution functions, and hence the conditional expectation is given by:

$$\begin{aligned}
\mathbb{E}\left[v\,|s\right] &= sF(s) + \int_s^1 yf(y)dy \\
&= s\left(s^{\frac{\alpha}{N+1}}\right) + \int_s^1 \frac{\alpha}{N+1} y^{\frac{\alpha}{N+1}} dy \\
&= \frac{N+1}{N+1+\alpha} s^{\frac{N+1+\alpha}{N+1}} + \frac{\alpha}{N+1+\alpha}.
\end{aligned}$$

The highest signal s among N bidders is therefore distributed according to $G(s)^{\frac{N}{N+1}}$ and the expected value conditional on highest signal is

$$v(s) \equiv \frac{N+1}{N+1+\alpha} s^{\frac{N+1+\alpha}{N+1}} + \frac{\alpha}{N+1+\alpha},$$

and inverting we have

$$s = \left(v\frac{N+1+\alpha}{N+1} - \frac{\alpha}{N+1}\right)^{\frac{N+1}{N+1+\alpha}} \tag{22}$$

The distribution of values v is thus given by:

$$H\left(v(s)\right) = G(s)^{\frac{N}{N+1}},$$

or using (22):

$$H(v) = \left(v\frac{N+1+\alpha}{N+1} - \frac{\alpha}{N+1}\right)^{\frac{N}{N+1+\alpha}},$$

with support

$$v \in \left[\frac{\alpha}{N+1+\alpha}, 1\right].$$

We can then compute the revenue of the optimal mechanism with N bidders as if the signals in the optimal mechanism were distributed according to

$$F(s) = H(v)^{\frac{1}{N}} = \left(v\frac{N+1+\alpha}{N+1} - \frac{\alpha}{N+1}\right)^{\frac{1}{N+1+\alpha}}. \tag{23}$$

We can then compute the revenue of the optimal mechanism using Theorem 1. The generalized virtual utility function $\psi(x)$ for the signal distribution given by (23) can be computed using integration by parts:

$$\begin{aligned}\psi(x) &= x - \int_x^1 \frac{1-F(y)}{F(y)}dy \\ &= \left(\frac{N+1+\alpha}{N+1}x - \frac{\alpha}{N+1}\right)^{\frac{N+1}{N+1+\alpha}} \end{aligned}\tag{24}$$

Thus, the generalized virtual utility is nonnegative for all values in the support of $F(v)$, and thus the optimal mechanism has no reserve price.

The expected revenue from the optimal mechanism is therefore given by (14):

$$\begin{aligned}R &= \int_{\frac{1}{N+1+\alpha}}^1 \psi(x)NF^{N-1}(x)f(x)dx \\ &= \frac{N+\alpha-1}{2N+\alpha}, \end{aligned}\tag{25}$$

where the second line follows from integration by parts after inserting (23) and (24).

We can then compare the revenue from the optimal mechanism with the revenue from the second price auction, and thus

$$\begin{aligned}&\frac{N+\alpha-1}{2N+\alpha} - \frac{\alpha^2}{\alpha+1}\frac{N}{(N+1)(1+\alpha)-\alpha} \\ =&\frac{2N^2\alpha + N^2 + N\alpha^2 + \alpha^2 - 1 - N^2\alpha^2}{(\alpha+1)(2N^2\alpha + 2N^2 + N\alpha^2 + N\alpha + 2N + \alpha)}\end{aligned}$$

which delivers the results as eventually, as α grows, $N^2\alpha^2$ becomes the dominant term. $\square$