Revenue sharing in second-price auctions

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Abstract

I consider a second-price sealed bid auction in which the seller asks losing bidders to recommend a reservation price for the high bidder. This mechanism is useful in environments where the seller does not know enough about the distribution of values to set the optimal reservation price, but the buyers do. Each bidder is incentivized to report the optimal reserve by giving them a small share of revenue when a sale is made using the suggested price. Revenue sharing aligns the incentives of the seller and losing buyers, but creates incentives to "throw" the auction when a buyer would expect to win at a price close to his valuation. For private value symmetric environments in which bidders' first-order beliefs satisfy a monotone hazard rate condition, I characterize a symmetric equilibrium in which bids and suggested prices are weakly increasing. As revenue sharing goes to zero, the distortion in bidding behavior disappears, and the auction implements the optimal reserve prices. I also discuss extensions to asymmetric distributions and more general type spaces.

Keywords: Second-price auction, revenue sharing, reservation price, shading, privacy.
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1 Introduction

The second-price auction with optimal reserve prices has many desirable properties. There is a compelling equilibrium in which bidders follow the unique weakly undominated strategy of bidding their values. In benchmark environments, the second-price auction with a judiciously chosen reserve price is an optimal auction (Myerson, 1981). Even for more general classes of environments, the second-price auction is a fair approximation of the optimal auction. Hartline and Roughgarden (2009) show that the second-price auction with an optimal anonymous reserve price attains at least 25% of the revenue of the optimal auction, as long as the distribution of values is independent and regular. With bidder specific reserves, this improves to 50% (Chawla et al., 2007). Also, it is known that if the seller sets an optimal reserve price for the winner conditional on the losers' values, the auction generates at least 50% of the maximum revenue possible in a dominant strategy and ex-post individually rational mechanism (Ronen, 2001).

The list goes on. The gist is that the second-price auction is a reliable mechanism for generating revenue, with the proviso that the seller must set the correct reserve price. This is no trivial matter. A large literature in auction econometrics has explored the two step process of estimating the distribution of values from bid data, and using this as an input to calculate an optimal reserve price (Athey and Haile (2007) survey this literature). An inescapable fact is that these methods require lots of data and/or non-trivial assumptions on the distribution of values in order to back out the distribution from bidding behavior.

In this paper, I explore an alternative route to the optimal reserve price: Ask the bidders. If the buyers are themselves well-informed about the distribution of values, the seller could elicit this information and have the bidders set reserve prices for one another. I consider a variation on the second-price auction in which the seller asks losing bidders to suggest reservation prices for the high valuation bidder. When a sale is made using a bidder's suggestion, that bidder receives a small share of the resulting revenue. This linear revenue-sharing contract perfectly aligns the incentives of the buyer and seller, conditional on the buyer losing. As such, this seems to be the simplest and most natural candidate for eliciting the optimal reserve.

However, in order to obtain a share of revenue, a bidder has to lose. Since a bidder will only lose the auction when bidding b when some other participant bids more, the suggested reserve price conditional upon losing must be at least b. I call this the pricing constraint. If the pricing constraint is binding, then at the margin, shared revenue would increase if the bidder were to shade his bid. On the other hand, the pivotal allocation that is affected by bidding one's value v is when the price paid would be close to the bidder's value as well. In this event, the surplus from receiving the good will be zero. Thus, players cannot bid their value if the pricing constraint binds. Of course, if the buyer were to shade a large amount so as to set a very low reserve price, then price at which revenue would be shared would be much lower than v. Clearly there would not be a benefit to shading close to zero. This suggests that there could be an equilibrium level of shading that is positive but not too large.

My main result is to show that for environments satisfying a particular positive correlation condition, there is indeed a simple equilibrium in which bidders shade to balance the marginal surplus they could obtain by being allocated the good and the marginal shared revenue they obtain by losing and suggesting reserve prices for others. The positive correlation condition takes the form of a requirement that the conditional hazard rate for the distribution of the highest value of others is monotonically decreasing in one's own value. This condition is similar in spirit to other positive correlation conditions used in the literature, such as the affiliated values of Milgrom and Weber (1982) or monotone likelihood ratios of Athey (2001). The equilibrium bidding function is characterized by regions on which the pricing constraint binds, where the bidding function solves a differential equation equating the sum of marginal surplus and marginal revenue with zero, and regions on which the pricing constraint does not bind and bidding one's value is a best response.

In equilibrium, a bidder with valuation v bids no less than $\frac{v}{1+\frac{\alpha}{n-1}}$, where α is the share of revenue when the suggested reserve price is used and n is the number of participants. Thus, as α goes to zero, shading disappears, bids converge to values, and the reserve prices converge to their optimal quantities. As a practical matter, there may be a point at which α is so small that bidders are not sensitive to the revenue sharing incentive. A nice feature of my results is that for strictly positive α, I have a tight characterization of an equilibrium that yields bounds on the revenue lost from sharing and distortions. In particular, the seller is always guaranteed a revenue of $\pi^* \frac{1-\alpha}{1+\frac{\alpha}{n-1}}$, where π^* is revenue from the second-price auction with an optimally chosen anonymous reserve price.

In addition to the benchmark model described above, I also consider several extensions. First, I discuss what happens with asymmetric distributions in the context of an independent two bidder example. While a symmetric distribution gives rise to strictly increasing bidding functions, with asymmetric bidders, there may be regions where one player's bidding function is constant and the other's is decreasing. Moreover, there may be regions of the bid space on which one bidder's pricing constraint is binding and the other bidder's constraint is slack. Second, in the benchmark mechanism described above, bidders are rewarded with a share of the seller's realized revenue, but only on the event that the bidder loses. A variation on this mechanism rewards each bidder i regardless of whether they win the auction, with a share of "simulated" revenue that would have obtained if the seller had run the $n-1$ bidder auction excluding bidder i and using his suggested reserve price. Since bidders are rewarded regardless of the allocation, there is no incentive to throw the auction, and truthful bidding is an equilibrium. For regular, symmetric, and independent distributions, this mechanism does just as well, but I discuss its limitations in more general environments. Finally, the assumptions of positive correlation and a one-dimensional structure on types are needed to demonstrate the existence of an equilibrium. If existence is assumed, then I give an example of a mechanism with a slightly less intuitive structure that nonetheless provides similar revenue bounds in general type spaces.

In related work, I have explored how an auction designer can extract details of the environment from well-informed buyers. Brooks (2013a) shows that the designer can effectively extract "for free" all of information that is common knowledge among the agents. The seller can use this information freely to design the mechanism, without affecting bidders' incentives to tell the truth. The caveat is that this mechanism requires the agents to report their entire hierarchy of beliefs, a decidedly complicated object. In contrast, Brooks (2013b) looks for simpler mechanisms that maximize the minimum extraction ratio, which is the ratio between expected revenue and the expected surplus that would be generated by allocating the good efficiently. This problem is solved by a modified second-price auction, in which bidders report bids as well as first-order beliefs about the distribution of others' values. The seller uses these first-order beliefs to calculate reserve prices. Both of these papers cover general finite type spaces, in which bidders can have any higher-order beliefs that are consistent with a common prior.

The present paper, in contrast, looks at more structured environments in which bidders have a single hierarchy of beliefs corresponding to each value and the joint distribution of values admits a density. In such environments, second-price auctions with revenue sharing allow the seller to elicit the optimal reserve prices with the simplest possible message space, consisting of bids and suggested prices. Granted, bidders are asked to compute an optimal reserve conditional on losing at their value, which is a non-trivial task. However, by sharing revenue, the losing bidders' incentives are closely aligned with those of the seller. In contrast, the mechanisms explored in Brooks (2013a,b) incentivize the bidders to report their beliefs using scoring rules, and then the seller performs the computation. It may well be that computing optimal or near optimal reserves is easier for the bidders than communicating a distribution. For example, the bidders may have privacy concerns with regard to their private information about their competitors. Reporting a price allows the bidders to communicate what the seller needs without divulging any extra information that might be a liability.

There is a small but growing literature on how the seller can run an auction and simultaneously calibrate auction parameters using ancillary reports made by the buyers about the environment. Caillaud and Robert (2005) consider how a seller can partially implement the optimal auction of Myerson (1981) through a dynamic mechanism. Dasgupta and Maskin (2000) construct a mechanism that partially implements the efficient outcome in interdependent value settings, in which bidders submit a function that gives a bid for every possible valuation of the other player. The seller computes the winner and price by looking for a fixed point of the reported mappings, and as such, the seller needs no additional information about the environment beyond what is reported. Azar et al. (2012) also study the use of scoring rules to recover a truncated prior distribution over values, and then use this prior to design the mechanism.

The rest of this paper is organized as follows. Section 2 describes the environment and defines the second-price auction with revenue sharing. Section 2 also gives a definition of my equilibrium concept, which imposes regularity conditions on the bidding function. Section 3 gives a simple example that illustrates some of the main ideas of my construction. Section 4 provides this general characterization for joint distributions of private values with positive correlation. Section 5 discusses several extensions, and Section 6 concludes.

2 Model

Some preliminary notation: For a vector x, let $x^{(1)}$ denote the highest value, $x^{(2)}$ the second-highest value, and $x^{(1,2)}$ the ordered pair of the highest and second-highest values. If x has a single coordinate, then $x^{(2)} = -\infty$. Let $W(x) = \{i | x_i = x^{(1)}\}$ denote the set of high value indices. I denote by $x \vee y$ and $x \wedge y$ the maximum and minimum of x and y, respectively. I also use the usual convention that x_S denotes the sub-vector of x with indices in S, and x_{-S} denotes the sub-vector with indices not in S.

There are n bidders, indexed by $i \in N = \{1, \dots, n\}$. Bidders have private valuations for a single unit of a good that are jointly distributed according to the cumulative distribution $F(v_1, \dots, v_n)$ with compact support $[\underline{v}, \overline{v}]^n$. This distribution is symmetric in v and admits a strictly positive and continuous density $f(v)$. In Section 5, I extend the analysis to a class of asymmetric distributions. I assume that bidders do not receive any additional information beyond their private value. This distribution, while unknown to the seller, is known to the bidders.

I will have need of several conditional densities and cumulative distributions, including but not limited to $f_{v_j|v_i}(\cdot|\cdot)$, $f_{v_{-i}^{(1)}|v_i}(\cdot|\cdot)$, $f_{v_{-ij}^{(1)}|v_i,v_j}(\cdot|\cdot,\cdot)$ where ij is shorthand for $\{i,j\}$. For reasons which will subsequently become clear, I assign compact notation to the following quantity:

$$g(x,y|v) = \frac{F_{v_{-ij}^{(1)}|v_i,v_j}(y|x,v) f_{v_j|v_i}(x|v)}{F_{v_j,v_{-ij}^{(1)}|v_i}(x,\overline{v}|v) - F_{v_j,v_{-ij}^{(1)}|v_i}(x,y|v)}.$$

This is the hazard rate of bidder j's value when values of bidders $k \neq i,j$ are less than y, conditional on bidder i's value being v. This quantity is closely related to the choice of an optimal reserve price if some bidder i with valuation v were to sell the good to the remaining $n-1$ bidders. With a reserve price r, revenue would be

$$\int_{x=r}^{\overline{v}} \left[r F_{v_{-ij}^{(1)}|v_i,v_j}(r|v,x) + \int_{y=r}^{x} y f_{v_{-ij}^{(1)}|v_i,v_j}(y|v,x) dy \right] f_{v_j|v_i}(x|v) dx, \tag{1}$$

where bidder j is taken to be a "representative" bidder in $-i$ with the highest value. Such a bidder pays r if $v_{ij}^{(1)} \leq r$, and pays $v_{-ij}^{(1)}$ otherwise. The derivative with respect to r is

$$\begin{aligned} &\int_{x=r}^{\overline{v}} F_{v_{-ij}^{(1)}|v_i,v_j}(r|v,x) f_{v_j|v_i}(x|v) dx - r F_{v_{-ij}^{(1)}|v_i,v_j}(r|v,r) f_{v_j|v_i}(r|v) \\ &\quad = \left(F_{v_j,v_{-ij}^{(1)}|v_i}(\overline{v},r|v) - F_{v_j,v_{-ij}^{(1)}|v_i}(r,r|v) \right) (1 - r\, g(r,r|v)) . \end{aligned} \tag{2}$$

I make the following two assumptions:

A1 For every x and y, the function $g(x,y|v)$ is weakly decreasing in v.

A2 There exists finitely many v at which $g(r,r|v) = \frac{1}{v}$.

A1 is a substantive restriction, analogous to the positive correlation conditions of Milgrom and

Weber (1982) and Athey (2001). It essentially requires that higher values of v_i make higher values of v_j more likely, in the sense that for every interval $[x, \overline{v}]$, v_j is less likely to be at the bottom of the interval x when $v_{-ij}^{(1)} \leq y$. **A1** is trivially satisfied in the case of independent values, and for two bidders it reduces to the familiar monotone hazard rate condition, that $\frac{f(r|v)}{1-F(r|v)}$ is decreasing in v. This property will ensure that higher valuation bidders want to set higher reserve prices in equilibrium. **A2** is a technical illustration which facilitates a simple equilibrium construction. Without **A2**, I would have to address the possibility that there are regions where $g(v,v|v) = \frac{1}{v}$, which in the independent private value setting correspond to cases where the virtual valuation is zero. Also, if there are infinitely many points at which this equality holds, transfinite induction would be required for the proofs of my main results, as opposed to the finite induction currelty used.

The seller of the good uses the following *revenue share auction* (RSA). Each bidder i submits a bid b_i and a reserve price r_i. If the profile of bids is b, then the seller picks a bidder $i \in W(b)$ uniformly, and then picks a losing bidder $j \neq i$ uniformly to consult for the price. If $b_i \geq r_j$, then bidder i is awarded the good at price $\max\left\{r_j, b_{-j}^{(2)}\right\}$, and bidder j receives $\alpha \max\left\{r_j, b_{-j}^{(2)}\right\}$ as his "share" of the revenue, with $\alpha \in (0,1]$. Otherwise, the good remains unallocated, and no transfers are made. The interpretation is that bidder j sets the reserve price in the $n-1$ bidder auction excluding j, and receives an α share of revenue generated by that auction.

I will define a class of symmetric equilibria consisting of a bidding function $\beta(v)$ and a pricing function $\rho(v)$, which are required to satisfy the following two properties:

E1 The function β is continuous and strictly increasing.

E2 For all $v \in [\underline{v}, \overline{v})$, one of the two holds:

(i) $\beta(v) = v$, or

(ii) A bidder of type v strictly prefers bidding $\beta(v)$ to bidding v.

In other words, each player bids their value unless there is a strict incentive to do otherwise. I exclude the type with $v_i = \overline{v}$ from this requirement, for reasons which will be seen shortly. Note that **E1** implies that β has a well defined inverse β^{-1} on its range. If $b > \beta(\overline{v})$, take $\beta^{-1}(b) = \overline{v}$, and similarly if $b < \beta(\underline{v})$, $\beta^{-1}(b) = \underline{v}$. In addition, the following incentive constraint must be satisfied:

$$S(v, \beta(v)) + R(v, \beta(v), \rho(v)) \geq S(v, b) + R(v, b, r) \tag{3}$$

for all (v, b, r), where

$$S(v,b) = \mathbb{E}\left[\left(v - \rho(v_j) \vee \beta\left(v_{-ij}^{(1)}\right)\right) \mathbb{I}_{\rho(v_j) \vee \beta\left(v_{ij}^{(1)}\right) < b} \middle| v_i = v\right] \tag{4}$$

$$R(v,b,r) = \alpha\, \mathbb{E}\left[\rho(v) \vee \beta\left(v_{-i}^{(2)}\right) \mathbb{I}_{\beta\left(v_{-i}^{(1)}\right) \geq b \vee r} \middle| v_i = v\right] \tag{5}$$

are respectively surplus from being allocated the good and revenue from selling to others, when other participants use the bidding function β and pricing function ρ. I will say that (β, ρ) constitute a *regular equilibrium* if they satisfy **E1**, **E2**, and (3).

My main result is that a regular equilibrium of the RSA exists. This equilibrium has an intuitive structure in which bidders sometimes shade in response to the revenue-sharing incentives.

3 A simple example

It is instructive to start with a simple example of the kind of equilibrium that I will construct. Let us suppose that there are two bidders whose values are distributed independently and uniformly between 0 and 1. As described above, each bidder submits a bid b_i and a price r_i. If $b_i > b_j$, then bidder i "wins" the auction, but only receives the good if $b_i > r_j$. In this case, bidder i pays r_j to the seller, and bidder j receives $\alpha\, r_j$.

In the undominated equilibrium of the second-price auction, bidders bid their values, i.e., $\beta(v) = v$. Let us investigate whether or not bidders could use such a strategy in the RSA with a strictly positive α. If so, the distribution of bids is uniform on $[0,1]$, meaning that the optimal reseve price unconditional on losing is 0.5. However, bidders only share revenue when they lose, and the distribution of the other bidder's bids conditional on losing with a bid of b is uniform on $[b,1]$. Naturally, it cannot be optimal to set a reserve price $r < b$. This observation is valid more generally: In any regular equilibrium, it must be that $\beta(v) \leq \rho(v)$. As such, it makes sense to impose the pricing constraint that $r \geq b$ and simply write $R(v,r)$ instead of $R(v,b,r)$.

Hence, in an equilibrium in which $\beta(v) = v$, it must be that $\rho(v) \in \arg\max_{r \geq v} r(1-r)$, so that $\rho(v) = 0.5$ for $v < 0.5$, and $\rho(v) = v$ for $v > 0.5$. For this to be an equilibrium, it must be that (3) is satisfied. Note that

$$
\begin{aligned}
S(v,b) &= (v-0.5)0.5\, \mathbb{I}_{b \geq 0.5} + \int_{x=0.5}^{b} (v-x)dx, \\
R(v,r) &= \alpha\, r(1-r).
\end{aligned}
$$

The bidder's goal is to maximize

$$
U(v,b,r) = S(v,b) + R(v,r),
$$

subject to $r \geq b$. For any deviation $b \geq 0.5$, it is optimal to set $r = b$. Hence, the equilibrium bid must satisfy the following first-order condition:

$$
\frac{\partial S(v,b)}{\partial b} + \frac{\partial R(v,b)}{\partial b}\bigg|_{b=\beta(v)} \geq 0. \tag{6}
$$

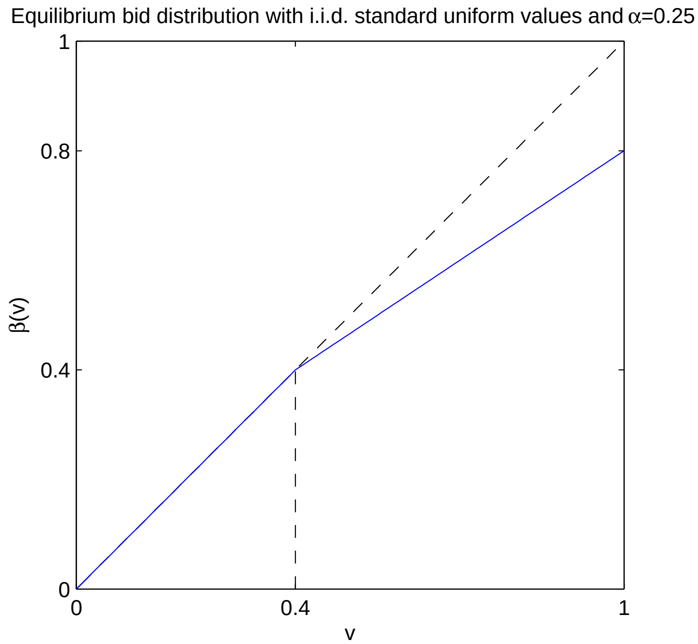

Figure 1: The equilibrium bidding function when values are distributed uniformly and independently on $[0, 1]$, and $\alpha = \frac{1}{4}$. Note that $v^* = 0.4$ and $\beta(1) = 0.8$.

Under the assumption that $\beta(v) = v$, this evaluates to

$$v - v + \alpha(1 - 2v) = 0,$$

which is obviously violated for $v > 0.5$. The intuition is as follows: The marginal allocation affected by shading when $b = v$ is when the other player sets a price $r = v$. In this case, the marginal surplus from the allocation is small, since $v - r \approx 0$. On the other hand, if $r > 0.5$, marginal revenue is strictly negative: If the constraint $r \geq b$ were not binding, a price of 0.5 would be optimal. At the margin, a bidder could shade a bit, and replace events on which he wins the good at a price close to v with events on which he sells at prices close to v, which leads to a strict improvement.

Indeed, there is a bidding function which does satisfy (6) when $r = \beta(v)$ is optimal, which is:

$$\beta(v) = \begin{cases} v & \text{if } v < v^* \\ \frac{\alpha + (1+\alpha)v}{(1+2\alpha)(1+\alpha)} & \text{if } v \geq v^* \end{cases}, \tag{7}$$

where $v^* = \frac{1}{2(1+\alpha)}$. This bidding function is depicted in Figure 1 for the case where $\alpha = \frac{1}{4}$. Note that the probability of a bid less than r is

$$F(r) = \begin{cases} r & \text{if } r < v^* \\ (1 + 2\alpha)r - \frac{\alpha}{1+\alpha} & \text{if } r \geq v^* \end{cases}. \tag{8}$$

As such, marginal revenue is

$$1 - F(r) - rf(r) = \begin{cases} 1 - 2r & \text{if } r < v^* \\ (1+2\alpha)\left(\frac{1}{1+\alpha} - 2r\right) & \text{if } r \geq v^* \end{cases},$$

which is clearly positive if $r < v^*$, and negative otherwise, as

$$\frac{1}{1+\alpha} - 2(1+2\alpha)r \leq -\frac{\alpha}{1+\alpha}.$$

As such, if $\beta(v)$ is an equilibrium bidding function, it must be that $\rho(v) = v^*$ if $v < v^*$ and $\rho(v) = \beta(v)$ otherwise. Consequently,

$$\begin{aligned} S(v,b) &= (v - v^*)v^* \, \mathbb{I}_{b \geq v^*} + \int_{x=v^*}^{b} (v - x)(1+2\alpha)dx, \\ R(v,r) &= \alpha \, r(1 - F(r)), \end{aligned}$$

where $F(r)$ is given by (8), and for $v > v^*$, (6) evaluates to

$$\begin{aligned} &\left(v - \frac{\alpha + (1+\alpha)v}{(1+\alpha)(1+2\alpha)}\right)(1+2\alpha) + \alpha(1+2\alpha)\left(\frac{1}{1+\alpha} - 2\frac{\alpha+(1+\alpha)v}{(1+\alpha)(1+2\alpha)}\right) \\ &= (1+2\alpha)\left(v + \frac{\alpha}{1+\alpha} - (1+2\alpha)\frac{\alpha+(1+\alpha)v}{(1+\alpha)(1+2\alpha)}\right) = 0. \end{aligned}$$

Hence, I conclude that (β, ρ) is indeed a regular equilibrium.

The form for β was not chosen arbitrarily. Note that $\beta(1) = \frac{1}{1+\alpha}$. In a regular equilibrium, the bidder with the highest valuation must win all the time with the highest bid, $\beta(1)$. The marginal surplus lost from shading is $(1 - \beta(1))f(1)$. On the other hand, shading makes it possible to sell to the other bidder when he has the highest value, the marginal revenue from which would be $-\alpha\beta(1)f(1)$. If the bidder with valuation 1 is indifferent to shading in equilibrium, then it must be that $\beta(1) = \frac{1}{1+\alpha}$. I will show in Lemma 2 that this condition generalizes to a requirement that in a regular equilibrium, $\beta(\overline{v}) = \frac{\overline{v}}{1+\frac{\alpha}{n-1}}$.

Also, suppose $\beta(v) = \rho(v)$ and is differentiable on a neighborhood of v in equilibrium, and consider local deviations to a bid $\beta(w)$ made by a nearby valuation w. In that case, the first order condition

$$\left.\frac{\partial S(v,\beta(w))}{\partial w} + \frac{\partial R(v,\beta(w))}{\partial w}\right|_{w=v} = 0$$

reduces to

$$(v - \beta(v))\beta'(v) + \alpha\left[(1-v)\beta'(v) - \beta(v)\right] = 0,$$

which can be rearranged to

$$\beta'(w) = \frac{\beta(w)(1+\alpha) - v}{\alpha} \frac{1}{1-v}.$$

One can guess that there is a linear solution in which $\beta'(v) = C$, and indeed there is, with $C = \frac{1}{1+2\alpha}$, which is precisely (7). Our equilibrium bidding function follows this differential equation until it hits v, at which point bidders are simply required to bid their values. I will show in Lemma 1 that this differential equation has a natural generalization to the framework of Section 2, and that this differential equation is a necessary condition whenever $\beta = \rho$ and β is differentiable on some neighborhood.

It is worth pointing out some nice features of this equilibrium. Given the explicit solution for β in (7), it is easy to see that as $\alpha \to 0$, $\beta(v) \to v$. In other words, the distortion created by revenue sharing is continuous in the amount shared, and with $\alpha = 0$ bids converge to the truthful bidding of the second-price auction. Moreover, for each α, bidders report the correct reserve price conditional on them losing with respect to the equilibrium bid distribution, which is $\frac{1}{2(1+\alpha)}$. Hence, as $\alpha \to 0$, the bid distribution converges to the value distribution, and the reserve price converges to $\frac{1}{2}$, and hence the seller is able to get close to revenue in the second-price auction with the optimally chosen reserve prices.

Finally, let us consider a slight variation of the linear example. Instead of uniform on $[0, 1]$, take the distribution of values to be uniform on $[\gamma - 1, \gamma]$, with $\gamma \geq 1$. The analogous differential equation is

$$\beta'(w) = \frac{\beta(w)(1+\alpha) - v}{\alpha} \frac{1}{\gamma - v},$$

which has the solution

$$\beta(v) = \frac{\gamma\alpha + v(1+\alpha)}{(1+\alpha)(1+2\alpha)}.$$

For γ sufficiently large, $\beta(\gamma - 1) < \gamma - 1$, so that the bidding function never leaves the regime with the binding pricing constraint. Moreover, as $\gamma \to \infty$, the ratio

$$\frac{\gamma - 1}{\beta(\gamma - 1)} \to 1 + \alpha,$$

so that in the limit, $\beta(v) \approx \frac{v}{1+\alpha}$. This result is intuitive: As γ becomes large, bids become large as well, but because values are compressed into the relatively small region $[\gamma - 1, \gamma]$, the bidders have to shade a large amount in order to obtain a large enough marginal surplus from winning to ofset the loss in marginal revenue. Nonetheless, the marginal surplus from winning in equilibrium is $v - \beta(v)$ and the marginal revenue from selling is $\alpha(\gamma - v - \beta(v)) \geq -\alpha\beta(v)$, so shading obeys the proportional bound of $\beta(v) \geq \frac{v}{1+\alpha}$.

4 A general symmetric equilibrium

In this Section, I will construct an equilibrium analogous to that of Section 3 for the general model of Section 2. In Section 4.1, I will investigate two necessary conditions of regular equilibrium, namely a boundary condition for the bid made by the highest valuation buyer, and a differential equation that must be satisfied when bids are equal to suggested prices. With these necessary conditions in hand, Section 4.2 describes an algorithmic construction of a bidding and pricing function. Section 4.3 gives a rich example that showcases features of the construction not appearing in the example of the previous section. Section 4.4 contains a summary of the proof that this strategy profile is indeed a regular equilibrium. Section 4.5 explores the revenue properties of the RSA, relative to the second-price auction with an optimal anonymous reserve price. All omitted proofs are in the Appendix.

4.1 Necessary conditions for regular equilibrium

To begin the analysis, I will prove that there are two necessary conditions for a regular equilibrium. The first is a generalization of the first-order condition (6), and the second is a boundary condition that is analogous to $\beta(1) = \frac{1}{1+\alpha}$ in the example of Section 3.

For starters, the choice of $\beta(v)$ effectively pins down $\rho(v)$, and more generally, it pins down the optimal price when bidding b. Conditional on bidder i losing with a bid of b when others are using the regular strategy (β, ρ), it must be that $\beta\left(v_{-i}^{(1)}\right) \geq b$. Each of the remaining bidders is equally likely to have the highest value among $-i$, so bidder j can be taken to be a "representative" high valuation player. Revenue is

$$R(v,b,r) = \alpha \int_{x=\beta^{-1}(b\vee r)}^{\overline{v}} \left[r\, F_{v_{-ij}^{(1)}|v_i,v_j} \left(\beta^{-1}(r) \middle| v, x \right) + \int_{y=\beta^{-1}(r)}^{x} \beta(y)\, f_{v_{-ij}^{(1)}|v_i,v_j} \left(\beta^{-1}(r) \middle| v, x \right) \right] f_{v_j|v_i}(x|v) dx. \tag{9}$$

Interpretation: Bidder i makes a sale if $\beta(v_j) \geq r$ and if $\beta(v_j) \geq b$ (since i has to lose the auction), which is the outer integral. Conditional on a particular realization for v_j, bidder i makes a sale at price r if $\beta(v_{-ij}^{(1)}) \leq r$, and makes a sale at price $\beta(v_{-ij}^{(1)})$ if $\beta(v_{-ij}^{(1)}) > r$. Since the lower limit for the first integral is $\beta^{-1}(b \vee r)$, it is never optimal to set a price less than b. Let us write

$$r^*(v,b) = \arg\max_{r \geq b} R(v,b,r).$$

Then an equilibrium condition is that $\rho(v) \in r^*(v, \beta(v))$. Note that continuity of β and f imply that $r^*(v,b)$ is non-empty and compact for all b, and upper-hemicontinuous in b. In general, the price $\rho(v)$ will fall into one of two cases: Either there is an interior maximum of $R(v, \beta(v), r)$ for $r \geq b$, in which case $\rho(v)$ is locally constant in v, or the maximizer is $r = \beta(v)$. When this second case obtains, and if $r^*(v,b) = \{b\}$, it will be the case that $\rho(v) = \beta(v)$ for a neighborhood around

$[b, b+\epsilon)$. In the following, I build in the fact that bidders would only choose $r \geq b$, in equilibrium or otherwise, and simply write $R(v, r)$.

Now consider the surplus that a bidder receives being allocated the good when bidding b. Note that this does not depend on the price that the bidder suggests.

$$S(v,b) = \int_{x=\underline{v}}^{\beta^{-1}(b)} \mathbb{I}_{b \geq \rho(x)} \Bigg[(v - \rho(x)) F_{v_{-ij}^{(1)}|v_i,v_j}(\beta^{-1}(\rho(x))|v,x) + \int_{y=\beta^{-1}(\rho(x))}^{\beta^{-1}(b)} (v - \beta(y)) f_{v_{-ij}^{(1)}|v_i,v_j}(y|v,x)dy \Bigg] f_{v_j|v_i}(x|v)dx \tag{10}$$

Here, I am using j as the index of the representative consulted bidder amongst $-i$, when bidder i wins the auction. A winning bidder i's total surplus is

$$U(v,b,r) = S(v,b) + R(v,r).$$

Suppose there is a neighborhood $(v-\epsilon, v+\epsilon)$ of v on which $r^*(w, \beta(w)) = \{\beta(w)\}$, so that $\rho(w) = \beta(w)$, and β and ρ are differentiable at v. Note that deviations to b near $\beta(v)$ can equivalently be thought of as deviations to $\beta(w)$ for nearby w, due to the continuity of β. Thus, the bidding function β must satisfy a first-order condition:

$$\left.\frac{dU(v,\beta(w),\beta(w))}{dw}\right|_{w=v} = \left.\frac{\partial S(v,\beta(w))}{\partial w}\right|_{w=v} + \left.\frac{\partial R(v,\beta(w))}{\partial w}\right|_{w=v} = 0. \tag{11}$$

This first-order condition can be translated into an equilibrium condition on β:

Lemma 1. *Suppose that there is a neighborhood $(v-\epsilon, v+\epsilon)$ of v on which $r^*(w, \beta(w)) = \{\beta(w)\}$, and β is differentiable at v. Then*

$$\beta'(v) = \frac{\beta(v)\,(1+\widehat{\alpha}) - v}{\widehat{\alpha}} g(v,v|v), \tag{FOC}$$

where $\widehat{\alpha} = \frac{\alpha}{n-1}$.

Proof of Lemma 1. Using the definition of S, and the fact that $\rho(\beta^{-1}(b)) = b$ for b in $(\beta(v-\epsilon), \beta(v+\epsilon))$, marginal surplus can be rewritten as

$$\frac{\partial S(v,\beta(w))}{\partial w} = \Bigg[(v-\beta(w)) F_{v_{-ij}^{(1)}|v_i,v_j}(w|v,w) f_{v_j|v_i}(w|v) + (v-\beta(w)) \int_{x=\underline{v}}^{w} f_{v_{-ij}^{(1)}|v_i,v_j}(w|v,x) f_{v_j|v_i}(x|v)dx \Bigg].$$

By symmetry, it must be that

$$\int_{x=\underline{v}}^{w} f_{v_{-ij}^{(1)}|v_i,v_j}(w|v,x) f_{v_j|v_i}(x|v)dx = (n-2) F_{v_{-ij}^{(1)}|v_i,v_j}(w|v,w) f_{v_j|v_i}(w|v),$$

so

$$\frac{\partial S(v,\beta(w))}{\partial w} = (n-1)(v-\beta(w))F_{v_{-ij}^{(1)}|v_i,v_j}(w|v,w)f_{v_j|v_i}(w|v).$$

The interpretation is that since $\beta = \rho$ around v, the marginal allocation event affected by bid $\beta(w)$ is when $v_{-i}^{(1)} = w$, which is the probability that one of the remaining bidders has a valuation of w and the other bidders have valuations less than w. Additionally, there are $n-1$ choices for the bidder with valuation exactly w.

The the second term in (11) is

$$\frac{\partial R(v,\beta(w))}{\partial w} = \alpha\Bigg[\left(F_{v_j,v_{-ij}^{(1)}|v_i}(\overline{v},w|v) - F_{v_j,v_{-ij}^{(1)}|v_i}(w,w|v)\right)\beta'(w) \\ - \beta(w)F_{v_{-ij}^{(1)}|v_i,v_j}(w|v,w)f_{v_j|v_i}(w|v)\Bigg],$$

with the other terms canceling or dropping out. Combining results, the marginal payoff is

$$\frac{\partial U(v,\beta(w),\beta(w))}{\partial w} = (n-1)\left(v-\beta(w)\left(1+\frac{\alpha}{n-1}\right)\right)F_{v_{-ij}^{(1)}|v_i,v_j}(w|v,w)f_{v_j|v_i}(w|v) \\ - \alpha\left(F_{v_j,v_{-ij}^{(1)}|v_i}(\overline{v},w|v) - F_{v_j,v_{-ij}^{(1)}|v_i}(w,w|v)\right)\beta'(w).$$

Now, evaluating at $w = v$ and rearranging yields equilibrium condition

$$\beta'(v) = \frac{\beta(v)\,(1+\widehat{\alpha}) - v}{\widehat{\alpha}} \frac{F_{v_{-ij}^{(1)}|v_i,v_j}(v|v,v)f_{v_j|v_i}(v|v)}{F_{v_j,v_{-ij}^{(1)}|v_i}(\overline{v},v|v) - F_{v_j,v_{-ij}^{(1)}|v_i}(v,v|v)},$$

where $\widehat{\alpha} = \frac{\alpha}{n-1}$. □

There might be a concern that $g(v,v|v)$ can blow up as $v \to \overline{v}$. However, one can prove directly that a solution to (FOC) subject to the boundary condition $\beta(\overline{v}) = \frac{\overline{v}}{1+\widehat{\alpha}}$ exists. If $\int_{x=\underline{v}}^{v} g(x,x|x)dx$ diverges as $v \to \overline{v}$, the solution is

$$\beta(v) = \frac{1}{\widehat{\alpha}}\exp\left(\frac{1+\widehat{\alpha}}{\widehat{\alpha}}\int_{x=\underline{v}}^{v} g(x,x|x)dx\right)\int_{x=v}^{\overline{v}}\exp\left(-\frac{1+\widehat{\alpha}}{\widehat{\alpha}}\int_{y=\underline{v}}^{x} g(y,y|y)dy\right)xg(x,x|x)dx. \quad (12)$$

Observe, the quantity

$$\int_{x=v}^{w}\exp\left(-\frac{1+\widehat{\alpha}}{\widehat{\alpha}}\int_{y=\underline{v}}^{x} g(y,y|y)dy\right)xg(x,x|x)dx$$

can be integrated by parts to give

$$-\frac{\widehat{\alpha}}{1+\widehat{\alpha}}\exp\left(-\frac{1+\widehat{\alpha}}{\widehat{\alpha}}\int_{y=\underline{v}}^{x} g(y,y|y)dy\right)x\Bigg]_{x=v}^{w}+\int_{x=v}^{w}\frac{\widehat{\alpha}}{1+\widehat{\alpha}}\exp\left(-\frac{1+\widehat{\alpha}}{\widehat{\alpha}}\int_{y=\underline{v}}^{x} g(y,y|y)dy\right)dx,$$

which converges to

$$\frac{\widehat{\alpha}}{1+\widehat{\alpha}}\exp\left(-\frac{1+\widehat{\alpha}}{\widehat{\alpha}}\int_{y=\underline{v}}^{v} g(y,y|y)dy\right)v+\int_{x=v}^{\overline{v}}\frac{\widehat{\alpha}}{1+\widehat{\alpha}}\exp\left(-\frac{1+\widehat{\alpha}}{\widehat{\alpha}}\int_{y=\underline{v}}^{x} g(y,y|y)dy\right)dx,$$

as $w \to \overline{v}$. Then taking $v \to \overline{v}$, this expression must converge to zero. We can then apply L'Hôpital's rule to (12) to find that

$$\lim_{v\to\overline{v}}\beta(v)=\lim_{v\to\overline{v}}\frac{1}{\widehat{\alpha}}\frac{-\exp\left(-\frac{1+\widehat{\alpha}}{\widehat{\alpha}}\int_{x=\underline{v}}^{v} g(x,x|x)dx\right)vg(v,v|v)}{-\frac{1+\widehat{\alpha}}{\widehat{\alpha}}g(v,v|v)\exp\left(-\frac{1+\widehat{\alpha}}{\widehat{\alpha}}\int_{x=\underline{v}}^{v} g(x,x|x)dx\right)}=\lim_{v\to\overline{v}}\frac{v}{1+\widehat{\alpha}}.$$

If $\int_{x=\underline{v}}^{\overline{v}} g(x,x|x)dx$ converges, then a term $C\int_{x=\underline{v}}^{\overline{v}} g(x,x|x)dx$ can be added so that the boundary condition obtains.

Now, consider the bidder with valuation $\overline{v}$. In a regular equilibrium, this type must make the largest bid $\beta(\overline{v})$. According to the rules described above, the bidder always wins and hence sells to other bidders with probability 0. In order for this to be incentive compatible, it must be that the type $\overline{v}$ does not have an incentive to shade, and start selling to bidders of lower valuation. At the margin, the surplus lost from not receiving the good when others bid $\beta(\overline{v})$ is $(v-\beta(\overline{v}))f_{v_{-i}^{(1)}|v_i}(\overline{v}|\overline{v})$, i.e. when one of the other bidders has a valuation of $\overline{v}$ conditional on $v_i=\overline{v}$. On the other hand, the revenue gained from selling to such a type is precisely $\widehat{\alpha}\beta(\overline{v})f_{v_{-i}^{(1)}|v_i}(\overline{v}|\overline{v})$. Hence, for shading not to be attractive for the highest type, it must be that $\beta(\overline{v})$ is less than $\frac{\overline{v}}{1+\widehat{\alpha}}$. In fact, if this were a strict inequality, then there is some type with value $v_i \in (\overline{v}-\epsilon,\overline{v}]$ who would prefer to shade less. This informal argument suggests the following Lemma, whose proof is in the Appendix.

Lemma 2. *In any regular equilibrium, it must be that*

$$\beta(\overline{v})=\frac{\overline{v}}{1+\widehat{\alpha}}, \tag{13}$$

where $\widehat{\alpha}=\frac{\alpha}{n-1}$.

4.2 A constructive algorithm

I will now construct a regular equilibrium of the RSA. The equilibrium consists of a partition of the interval of valations $[\underline{v},\overline{v}]$ into a sequence of intervals with endpoints

$$\underline{v}=\underline{w}^K\le\overline{w}^K\le\cdots\le\underline{w}^0\le\overline{w}^0=\overline{v}.$$

The partition, and the bidding and pricing functions β and ρ, will be defined inductively on the regions

$$\begin{aligned}\overline{W}^k &= (\underline{w}^k, \overline{w}^k] \\ \underline{W}^k &= (\overline{w}^{k+1}, \underline{w}^k].\end{aligned} \tag{14}$$

On regions $\overline{W}^k$, I set $\beta(v) = \rho(v)$ where β solves (FOC) with the initial condition $\beta(\overline{w}^0) = \frac{\overline{v}}{1+\widehat{\alpha}}$ and $\beta(\overline{w}^k) = \overline{w}^k$ for $k > 1$. On regions of the form $(\overline{w}^{k-1}, \underline{w}^k]$, I set $\beta(v) = v$ and $\rho(v) = r^*(v) = \inf r^*(v, v) \leq \underline{w}^k$.

In particular, let $\beta_k(v)$ be the solution to (FOC) on $[\underline{v}, \overline{w}^k]$ with the boundary condition

$$\beta_k(\overline{w}^k) = \begin{cases} \overline{w}^k & \text{if } k > 0 \\ \frac{\overline{v}}{1+\widehat{\alpha}} & \text{if } k = 0 \end{cases}. \tag{BC}$$

Define

$$\underline{w}^k = \sup\Big(\{\underline{v}\} \cup \{v < \overline{w}^k | \beta_k(v) > v\}\Big). \tag{15}$$

Note that this definition implies that if $\beta_k(v) \leq v$ for all v, then $\underline{w}^k = \underline{v}$. I define $\beta(v) = \beta_k(v)$ for all $v \in [\underline{w}^k, \overline{w}^k]$.

If $\underline{w}^k > \underline{v}$, let

$$\overline{w}^{k+1} = \sup\left(\{\underline{v}\} \cup \left\{v < \underline{w}^k \,\middle|\, R(v,v) > R(v,w)\ \forall w \in (v, \underline{w}^k]\right\}\right). \tag{16}$$

If $\underline{w}^k \in r^*(v, v)$ for all $v < \underline{w}^k$, then set $\overline{w}^{k+1} = \underline{v}$. Otherwise, $\beta(v) = v$ and $\rho(v) = r^*(v)$ for $v \in \underline{W}^k$.

The construction starts with $\overline{w}^0 = \overline{v}$, and continues inductively alternating between defining new $\underline{w}^k$ and $\overline{w}^{k+1}$. The algorithm terminates when the next of these two suprema are $\underline{v}$. This is formalized in Algorithm 1. Proposition 1 provides a characterization of the algorithm.

Proposition 1. *The inductive construction of Algorithm 1 terminates after finitely many steps. It defines a continuous and strictly increasing bidding function.*

Hence, the algorithm defines a bidding function. It is easy to see that this bidding function is continuous, since it either solves the differential equation (FOC) or is $\beta(v) = v$, and I have defined β at boundary points so that it is continuous. At this point, it is not known that β is strictly increasing, but this will follow from Lemma 3 below.

4.3 A more complicated example

Before showing that Algorithm 1 defines an equilibrium, let us look at an example that showcases the richness of the construction. The uniform example was relatively simple because Algorithm 1 converged after just two steps, which is a consequence of the fact that the value distribution has

Algorithm 1: Constructing a regular equilibrium

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initialize $k = 0$, $\overline{w}^0 = \overline{v}$.
initialize $\beta(v) = \rho(v) = \beta_0(v)$, which solves (FOC) and (BC)
while true
  define $\underline{w}^k$ according to (15).
  redefine $\beta(v) = \rho(v) = \beta_k(v)$ for $v \in [0, \underline{w}^k]$, where $\beta_k(v)$ solves (FOC) and (BC).
  if $\underline{w}^k = \underline{v}$,
    break.
  define $\overline{w}^{k+1}$ according to (15).
  redefine $\beta(v) = v$ and $\rho(v) = r^*(v)$ for $v \in [0, \underline{w}^k]$.
  if $\underline{w}^k = \underline{v}$,
    break.
  redefine k=k+1.
end while

---

monotonic virtual valuation, i.e., is regular in the sense of Myerson (1981). It also turned out that the bidding function on $\overline{W}^0$ had a simple linear form. Here I present a more complicated example involving two bidders in which values are still independent, but the independent distribution is highly irregular. The cumulative distribution of values is a weighted sum of Beta distributions, and in particular each bidder's valuation is distributed $B[\alpha = 1.5, \beta = 5.5]$ with probability 0.9 and is distributed $B[\alpha = 25, \beta = 2]$ with probability 0.1. The revenue sharing parameter is $\alpha = \frac{1}{4}$. This cumulate distribution results in the revenue curve $v(1 - F(v))$ depicted in Figure 2.

The equilibrium bidding function is depicted in Figure 3. Note that the solid line, $\beta(v)$, is everywhere above the dotted line, which is $\frac{v}{1+\alpha}$. The algorithm takes four regime changes to converge. It starts with $\overline{w}^0 = 1$ and $\beta(\overline{w}^0) = \frac{1}{1+\alpha} = 0.8$. Initially, β solves the differential equation (FOC) starting at $v = 1$ and going downwards, until $\beta(v)$ hits v at $\underline{w}^0 \approx 0.66$. At this point, the regime switches to $\beta(v) = v$ and $r^*(v) = 0.66$, until $\overline{w}^1 \approx 0.52$. At this point, r^* jumps down to 0.52, and the regime switches back to solving (FOC) with the boundary condition $\beta(\overline{w}^1) = \overline{w}^1$. The bidding function again hits v at $\underline{w}^1 \approx 0.18$, and the regime switches back to the $\beta(v) = v$ with $r^*(v) = 0.18$, until v hits zero.

In general, the algorithm could require many regime changes before reaching zero, although the number of regime changes is bounded above by two times the number of zeros of $g(v, v|v) - \frac{1}{v}$, as shown in the proof of Proposition 1.

4.4 The algorithm defines an equilibrium

My main result is the following:

Theorem 1. *The bidding and pricing functions (β, ρ) defined by Algorithm 1 constitute a regular equilibrium of the revenue-sharing second-price auction.*

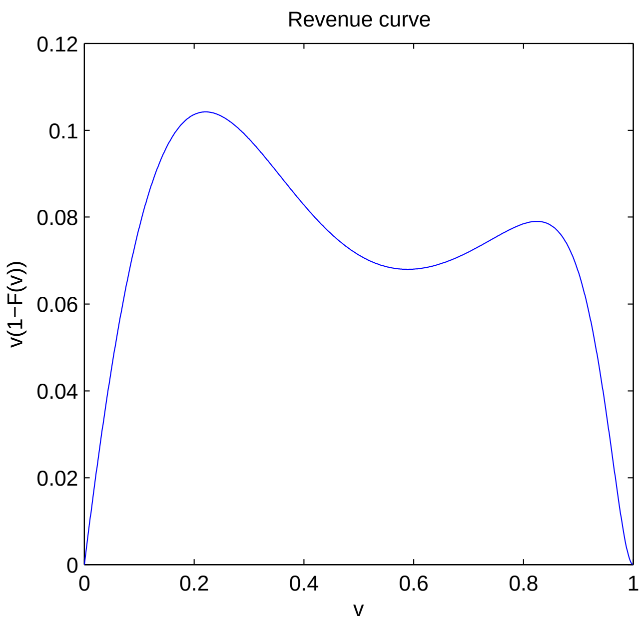

Figure 2: The revenue curve for the irregular example of Section 4.3. Note that the profit function has two peaks.

I will provide a general overview of the proof. To start, observe that (FOC) is zero when $\beta(v) = \frac{v}{1+\widehat{\alpha}}$. Hence, it is impossible for β to fall below this level, and indeed it is impossible for it to remain at this level for an open interval. This means that the bidding function is always strictly increasing, so if it is an equilibrium, it will be regular. This is formalized in Lemma 3.

Lemma 3. $\frac{v}{1+\widehat{\alpha}} \leq \beta(v) \leq v$.

An observation which greatly simplifies the proof is that there is a relatively small number of deviations which need to be checked. In particular, a deviation to (r, r) with $r < v$ dominates all deviations of the form (b, r) with $b < r < v$. The reason is that for $b < v$, $S(v, b)$ is weakly increasing, so it is without loss of generality to take b as large as possible subject to the pricing constraint. On the other side, only deviations of the form (v, r) where $r > v$ need to be considered. The reason is the same: $S(v, b)$ is weakly decreasing when $b > v$.

The next Lemma will help rule out some downward deviations. Recall that (FOC) defines the bidding function on regions $\overline{W}^k$. The first-order condition was obtained by differentiating $U(v, \beta(w), \beta(w))$ with respect to v and setting it equal to zero for $w = v$. However, substituting the definition of $\beta'(w)$ into the derivative of U yields the expression

$$\frac{\partial U(v, \beta(w), \beta(w))}{\partial w} = C(v, w) \cdot \Big[(\beta(w)(1+\widehat{\alpha}) - w)g(w, w|w) - (\beta(w)(1+\widehat{\alpha}) - v)g(w, w|v)\Big],$$

where $C(v, w)$ is some strictly positive number that depends on v and w. But because of Lemma 3, the term multiplying $g(w, w|w)$ is always non-negative, and also $\beta(w)(1+\widehat{\alpha}) - v$ is greater (less) than $\beta(w)(1+\widehat{\alpha}) - w$ if v is less (greater) than w. Combined with the fact that $g(w, w|v)$ is

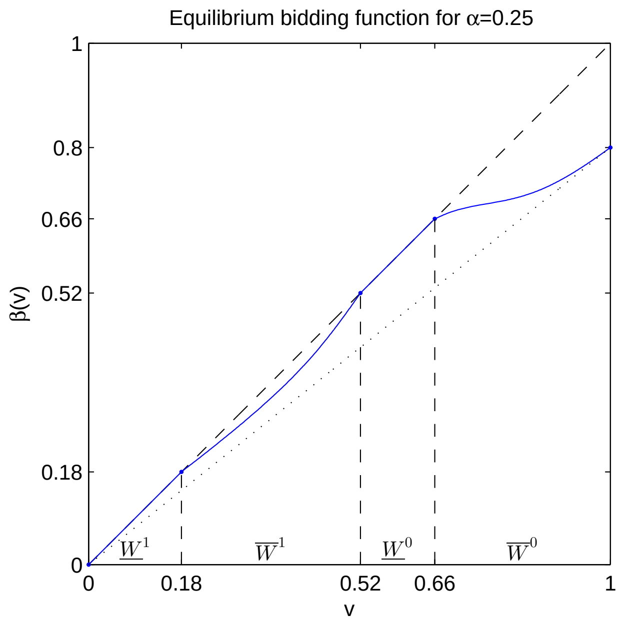

Figure 3: The equilibrium bid distribution for the example of Section 4.3. Algorithm 1 takes four steps to converge, with $\underline{w}^0 \approx 0.66$, $\overline{w}^1 \approx 0.52$, $\underline{w}^1 \approx 0.18$, and $\overline{w}^2 = 0$. The dotted line is $\frac{v}{1+\hat{\alpha}}$.

monotonically decreasing, these observations imply that the bidder's deviation payoff at a deviation of the form (r, r) is increasing if $b < \beta(v)$ and decreasing if $b > \beta(v)$.

Lemma 4. *For all w on the interior of $\overline{W}^k$,*

$$\frac{dU(v, \beta(w), \beta(w))}{dw} \begin{cases} \leq 0 & \text{if } w > v \\ \geq 0 & \text{if } w < v \end{cases} .$$

This Lemma tells us that a bidder's payoff is always decreasing as a deviation of the form (r, r) moves away from $(\beta(v), \rho(v))$, when the deviation bid is in a region on which the bidding function solves (FOC).

But to rule out large deviations, it must be that deviation payoffs are decreasing when crossing regions of the form $\underline{W}^k$, and also when deviating to (v, r) with $r > v$. This is facillitated by the following Lemma 5.

Lemma 5.

1. *For $r \in \underline{W}^k$, $\frac{\partial R(v,r)}{\partial r}$ is increasing in v. As a result, if $r \geq r'$ and $v \geq v'$, then*

$$v < \overline{w}^{k+1} \implies R(v, r) \leq R(v, \overline{w}^{k+1})$$
$$v > \overline{w}^{k+1} \implies R(v, r) \leq R(v, r^*(v) \wedge \underline{w}^k).$$

2. *For $w \in \overline{W}^k$ and $w \geq v$, $\frac{dR(v,\beta(w))}{dw} \leq 0$.*

The Lemma makes two assertions. The first concerns regions of the form $\underline{W}^k$, and asserts that if $v < \overline{w}^{k+1}$, then $\overline{w}^{k+1}$ generates greater expected revenue than any price $r \in \underline{W}^k$. Note that the result holds trivially when $\overline{w}^{k+1} = \underline{v}$. On the other hand, if $v > \overline{w}^{k+1}$, then either (1) $v \in \underline{W}^k$, and $r^*(v)$ is better than any price in $\underline{W}^k$, or (2) $v \notin \underline{W}^k$ and $\underline{w}^k$ is a better price than any $r \in \underline{w}^k$. Note that this is trivially satisfied when $\overline{w}^k = \overline{v}$. More generally, the result is a consequence of the positive correlation assumption **A1**, which is that higher types are more optimistic about the distribution of others' values. As a result, higher valuations always want to set higher reserve prices for other bidders.

The second part of Lemma 5 concerns the sign of marginal revenue on regions $\overline{W}^k$ with $v < \underline{w}^k$. By substituting in the formula for $\beta'(w)$, the derivative of U is

$$\frac{\partial U(v, \beta(w))}{\partial w} = C(v, w) \cdot \left[\frac{\beta(w) - w}{\widehat{\alpha}} g(w, w|w) + \beta(w)(g(w, w|w) - g(w, w|v)) \right],$$

where $C(v, w)$ is strictly positive. Since $\beta(w) \leq w$, and $g(w, w|w) < g(w, w|v)$, again it is the case that marginal revenue is non-positive on such regions.

The results of Lemmas 4 and 5 facilitate an inductive argument that the equilibrium strategy is optimal. For simplicity, let us consider $v \in \overline{W}^k$ for some k. The case when $v \in \underline{W}^k$ is not substantially different. The quasiconcavity of $U(v, \beta(w), \beta(w))$ means that there are no deviations of the form (r, r) that are optimal with $r \in \beta(\overline{W}^k)$. In particular, this means that $\underline{w}^k$ is not a profitable deviation. But then Lemma 5 implies that there is no profitable deviation on $[\overline{w}^{k+1}, \underline{w}^k] = \underline{W}^k$, since $R(v, \underline{w}^k)$ is greater than $R(v, r)$ for $r \in \underline{W}^k$ and $S(v, b)$ is weakly decreasing when $b < v$. Hence, a deviation to $(\overline{w}^{k+1}, \overline{w}^{k+1})$ is not profitable. But now the quasiconcavity kicks in again on $\overline{W}^{k+1}$, and there are no profitable deviations here either. This induction continues, and so that there are no profitable downward deviations.

With regard to upward deviations, it has already been shown that (v, v) is not a profitable deviation, since either (1) $v > \beta(\overline{v})$, in which case this is obvious, or (2) since $\beta(\overline{w}^k) \geq v$, (v, v) is a downward deviation in $\overline{W}^k$, which is not profitable because of Lemma 4. But now part 2 of Lemma 5 can be used to show that the payoff at (v, v) is weakly greater than the profit at (v, r) for all $r > v$. On regions $\overline{W}^k$, marginal revenue is non-positive because of Lemma 5, so $(v, \beta(\overline{w}^k))$ is not profitable. If $k > 0$, the first part of Lemma 5 shows that $(v, \overline{w}^k)$ is better than any deviation (v, r) with $r \in \underline{W}^k$. The induction continues, showing that no upward deviation is profitable. This concludes the proof sketch that (β, ρ) constitute a regular equilibrium of the RSA.

4.5 Equilibrium net revenue

Let us now turn our attention to revenue properties of the RSA, specifically with an interest in comparative statics as as $\alpha \to 0$. My basis for comparison is revenue from the second-price auction if the seller knew the distribution of values and was able to set the optimal anonymous reserve price. Formally, define

$$\pi^* = \max_{r \geq \underline{v}} \mathbb{E}\left[r\,\mathbb{I}_{v^{(2)} \leq r \leq v^{(1)}} + v^{(2)}\mathbb{I}_{r \leq v^{(2)}}\right],$$

and define r^* to be the revenue maximizing r, which is the optimal anonymous reserve price. Gross revenue from the RSA is

$$\pi^G = \int_{v=\underline{v}}^{\overline{v}} \mathbb{E}\left[\rho(v)\,\mathbb{I}_{\beta\left(v_{-i}^{(2)}\right) \leq \rho(v) \leq \beta\left(v_{-i}^{(1)}\right)} + v_{-i}^{(2)}\mathbb{I}_{\rho(v) \leq \beta\left(v_{-i}^{(2)}\right)} \,\middle|\, v_{-i}^{(1)} \geq v\right]\left(1 - F_{v_{-i}^{(1)}|v_i}(v|v)\right) f_{v_i}(v)dv,$$

and net revenue from the RSA is $\pi = (1-\alpha)\pi^G$, since an α share of revenue is awarded to the bidder who suggests the reserve price.

I will prove the following result:

Proposition 2. *For any $\alpha > 0$, net revenue π from the RSA under the equilibrium defined by Algorithm 1 is at least $\pi^* \frac{1-\alpha}{1+\widehat{\alpha}}$. Hence, as $\alpha \to 0$, revenue converges to a limit weakly greater than π^*.*

Proof of Proposition 2. Because of Lemma 3, we know that $\beta(v) \geq \frac{v}{1+\widehat{\alpha}}$. This implies that

$$\begin{aligned}
&\mathbb{E}\left[\frac{r^*}{1+\widehat{\alpha}}\,\mathbb{I}_{\beta\left(v^{(2)}\right) \leq \frac{r^*}{1+\widehat{\alpha}} \leq \beta\left(v^{(1)}\right)} + \beta\left(v^{(2)}\right)\mathbb{I}_{\frac{r^*}{1+\widehat{\alpha}} \leq \beta\left(v^{(2)}\right)}\right] \\
&\geq \mathbb{E}\left[\frac{r^*}{1+\widehat{\alpha}}\,\mathbb{I}_{v^{(2)} \leq r^* \leq v^{(1)}} + \frac{v^{(2)}}{1+\widehat{\alpha}}\,\mathbb{I}_{r^* \leq v^{(2)}}\right] \\
&= \frac{\pi^*}{1+\widehat{\alpha}}.
\end{aligned}$$

Hence, if the seller were to use the anonymous reserve price $\frac{r^*}{1+\widehat{\alpha}}$ with the equilibrium bid distribution induced by β, gross revenue would be at least $\frac{r^*}{1+\widehat{\alpha}}$.

In fact, the seller does not set the reserve price $\frac{r^*}{1+\widehat{\alpha}}$, but rather the reserve price $\rho(v)$ of a losing bidder with valuation v. However, each such bidder is setting an optimal reserve price conditional on $v^{(1)} \geq v$, and therefore is setting a reserve price which generates weakly greater expected revenue conditional on this event. Formally, gross revenue when using a particular bidder's recommendation is

$$\max_{r \geq \underline{v}} \mathbb{E}\left[r\,\mathbb{I}_{\beta\left(v_{-i}^{(2)}\right) \leq r \leq \beta\left(v_{-i}^{(1)}\right)} + v_{-i}^{(2)}\mathbb{I}_{r \leq \beta\left(v_{-i}^{(2)}\right)} \,\middle|\, v_{-i}^{(1)} \geq v\right].$$

Hence, gross revenue is

$$\begin{aligned}
&\int_{v=\underline{v}}^{\overline{v}} \max_{r\geq \underline{v}} \mathbb{E}\left[r\,\mathbb{I}_{\beta\left(v_{-i}^{(2)}\right)\leq r\leq \beta\left(v_{-i}^{(1)}\right)} + v_{-i}^{(2)}\mathbb{I}_{r\leq \beta\left(v_{-i}^{(2)}\right)} \middle| v_{-i}^{(1)} \geq v \right] \left(1 - F_{v_{-i}^{(1)}|v_i}(v|v)\right) f_{v_i}(v)dv \\
&= \int_{v=\underline{v}}^{\overline{v}} \max_{r\geq \underline{v}} \mathbb{E}\left[r\,\mathbb{I}_{\beta\left(v^{(2)}\right)\leq r\leq \beta\left(v^{(1)}\right)} + v^{(2)}\mathbb{I}_{r\leq \beta\left(v^{(2)}\right)} \middle| v_{-i}^{(1)} \geq v \right] \left(1 - F_{v_{-i}^{(1)}|v_i}(v|v)\right) f_{v_i}(v)dv \\
&\geq \max_{r\geq \underline{v}} \int_{v=\underline{v}}^{\overline{v}} \mathbb{E}\left[r\,\mathbb{I}_{\beta\left(v^{(2)}\right)\leq r\leq \beta\left(v^{(1)}\right)} + v^{(2)}\mathbb{I}_{r\leq \beta\left(v^{(2)}\right)} \middle| v_{-i}^{(1)} \geq v \right] \left(1 - F_{v_{-i}^{(1)}|v_i}(v|v)\right) f_{v_i}(v)dv \\
&= \max_{r\geq \underline{v}} \mathbb{E}\left[r\,\mathbb{I}_{\beta\left(v^{(2)}\right)\leq r\leq \beta\left(v^{(1)}\right)} + v^{(2)}\mathbb{I}_{r\leq \beta\left(v^{(2)}\right)} \right].
\end{aligned}$$

The second line comes from the fact that $r \geq \beta(v)$, $v_{-i}^{(2)} \geq v^{(2)}$, and $v_{-i}^{(1)} = v^{(1)}$. The third line comes from the integral of the maximum being greater than the maximum of the integral. The final line is just the law of iterated expectations. The last line is at least $\frac{\pi^*}{1+\widehat{\alpha}}$, so gross revenue under the RSA is at least this quantity as well.

However, the seller is also making payments to the agents of α times realized revenue. Hence, net revenue is $1-\alpha$ times gross revenue. □

Thus, the loss from revenue sharing becomes small as $\alpha \to 0$. This result is intuitive: For any α, bidders suggest optimal reserve prices conditional on them losing the auction, with respect to the equilibrium bid distribution. Since these prices are optimal conditional on more information than the prior, namely the realization of the loser's value v and the fact that $v_{-i}^{(1)} \geq v$, revenue generated with such prices is at least the revenue with an optimal ex-ante reserve prices. Moreover, because of Lemma 3, as $\alpha \to 0$ the equilibrium bid distribution converges to the distribution of values, and since revenue is continuous in the distribution of values, gross revenue converges to a quantity weakly greater than π^*. Lastly, for small α, the revenue lost from sharing is small relative to gross revenue.

In light of Proposition 2, it is fair to say that the RSA accomplishes the goal described in the introduction, which is to approximate revenue from the second-price auction with an optimal anonymous reserve, even in situations where the seller does not know the distribution of values but the buyers do.

5 Discussion

5.1 Asymmetric distributions

Throughout the analysis, I have restricted attention to symmetric case. The extension to asymmetric distribution involves a somewhat more complicated construction than Algorithm 1, and some new conceptual challenges. In the symmetric case, bidders all used the same bidding function and hence at any valuation v, all bidders were either in the regime determined by the first-order condition (FOC) or were bidding their values. With asymmetric bidders and asymmetric bidding functions, the bidders' regimes need not coincide, and indeed I must make allowance for one bidder to be following the asymmetric version of (FOC) and the other bidder to bid his value.

To illustrate, let us consider a two bidder example in which each bidder i's value is drawn independently from the distribution with cumulative distribution F_i. I assume that both F_i have the same support $[\underline{v}, \overline{v}]$ and both admit strictly positive and continuous densities f_i. I will first derive the asymmetric analog of (FOC). To that end, it is useful to define the functions

$$z_i(b) = \beta_i^{-1}(b),$$

which are the inverse bid functions. In the asymmetric case, I will solve directly for the inverse functions, and then invert them to obtain bidding functions.

In that case, surplus and revenue can written

$$\begin{aligned} S_i(v,b) &= \int_{x=\beta(\underline{v})}^{b} (v - \rho_j(z_j(x))) f_j(z_j(x)) z_j'(x) dx, \\ R_i(v,r) &= \alpha\, r \left(1 - F_j\left(z_j(r)\right)\right). \end{aligned}$$

Hence, the condition that bidder i's marginal surplus plus marginal revenue equal zero reduces to

$$(v-b) f_j(z_j(b)) z_j'(b) + \alpha \left[1 - F_j(z_j(b)) - b\, f_j(z_j(b)) z_j'(b)\right] = 0,$$

which evaluated at $v = z_i(b)$ can be rewritten as

$$z_j'(b) = \frac{\alpha}{b(1+\alpha) - z_i(b)} \frac{1 - F_j(z_j(b))}{f_j(z_j(b))}. \tag{FOC$'$}$$

This formula has an important feature missing from the symmetric case: The first-order condition for bidder i to price at his bid is actually a constraint on bidder j's bidding function. The boundary condition is unchanged:

$$z_j\left(\frac{\overline{v}}{1+\alpha}\right) = \overline{v}. \tag{BC$'$}$$

Our new algorithm again calls for initially solving (FOC$'$) subject to (BC$'$). The construction starts with $z_i(b) \geq b$, and the regime switches when some $z_i(b) - b$ hits 0. With symmetric distributions, both z_i would hit b at the same time (if at all). However, we must confront the possibility that $z_i(b) - b$ hits 0 at $\widehat{b}$, but $z_j(b) - b > 0$ for all $b \geq \widehat{b}$.

One idea would be to look for hybrid regimes in which one player's z_i is defined using the first-order condition, and the other player has $z_j(b) = b$. However, this cannot be part of an equilibrium. For suppose that this is the case, say with z_1 solving (FOC$'$) and $z_2(b) = b$. This would imply that bidder 1 is pricing strictly above his bid, and bidder 2 is pricing at his bid on some region. But this requires that $z_1(b) > b$, in which case bidder 1 is shading, even though his pricing constraint is not binding and there is positive probability of bidder 1 setting a price of b' between b and $z_1(b)$.

As such, bidder 1 would want to increase his bid, so as to win on these events!

Therefore, a hybrid regime cannot exist when leaving a regime where both pricing constraints bind. But, it still might be the case that the solution of $z_2(b) - b$, say, hits 0 at $\widehat{b}$, while $z_1(\widehat{b}) > \widehat{b}$. What then? If z_1 is to be monotonic, the only option is to have z_1 jump down to $z_1(b) = b$, so that z_1 has a discontinuity. This corresponds to a range of valuations for player 1, between $z_1^+(\widehat{b})$ and $z_1^-(\widehat{b})$, limits from the right and left respectively, who all bid $\widehat{b}$ and set a price of $\widehat{b}$. Intuitively, these types all want to sell to a bidder 2 with value greater than $\widehat{b}$, and bidder 2 bids his value and sets a price of $\widehat{b}$, effectively selling to the mass point.

To illustrate, let us solve a simple asymmetric example. The support of values is $[0, 1]$, and $F_1(x) = x$ and $F_2(x) = x^2$. Hence, bidder 2 is the "high demand" consumer, with expected valuation of $\frac{2}{3}$, whereas bidder 1's expected value is $\frac{1}{2}$. The differential equations are

$$z_1'(b) = \frac{\alpha}{b(1+\alpha) - z_2(b)}(1 - z_1(b)), \tag{17a}$$

$$z_2'(b) = \frac{\alpha}{b(1+\alpha) - z_1(b)} \frac{1 - (z_2(b))^2}{2z_2(b)}. \tag{17b}$$

The construction starts with $\widehat{b}^0 = \frac{1}{1+\alpha}$ and $\omega^0 = (B, B)$. It turns out that $z_2(b) - b$ hits 0 first, at around $\widehat{b} \approx 0.4430$, and thereafter set $z_1(b) = z_2(b) = b$. This is an equilibrium, since a price of $\widehat{b}$ dominates all lower bids. This can be seen from the fact that $v(1 - F_1(v))$ is concave with a maximum at $v = \frac{1}{2}$, and $v(1 - F_2(v))$ is concave with a maximum at $v = \frac{1}{\sqrt{3}} \approx 0.5774$.

What if the z_i are in the regime where $z_1(b) = z_2(b) = b$, and then at some $\widehat{b}$ bidder 1's pricing constraint binds, so that he would want to start shading in equilibrium? In order to satisfy bidder 1's indifference while maintaining $z_1(b) = b$, bidder 2 would need to start shading. However, this shading cannot be incentive compatible if bidder 1 is setting prices between b and $z_2(b)$ with positive probability. The solution is to solve (FOC$'$) with $z_2'(b) = 1$, so that

$$(b(1+\alpha) - z_1(b))f_2(b) = \alpha(1 - F_2(b)),$$

until bidder 2's pricing constraint binds, at which point the z_i solve the full system of first-order conditions.

Note that the difference between leaving the both-not-binding regime considered here and leaving the both-binding regime considered above is that the player who continues to bid his value must have a weak incentive to price above his own value. When leaving the both-binding regime, when $z_i(b) - b$ hits 0 at $\widehat{b}$, this means that bidder j now has an incentive to price at $\widehat{b}$, and hence he cannot shade to a bid below $\widehat{b}$. On the other hand, when leaving the both-not-binding regime, there is no problem having one bidder continue to bid his value as long as he prices above his value, while the other bidder starts to shade.

Thus, the general lessons for the two bidder asymmetric case are

1. When leaving a regime with both z_1 and z_2 solving the first-order condition, and when $z_i(b) - b$

hits 0 first at $\widehat{b}$, then $z_j(b)$ jumps down, so that β_j is constant at $\widehat{b}$ until $v = \widehat{b}$.

2. When leaving a regime with $z_1(b) = z_2(b) = b$, and bidder i's pricing constraint binds first, then bidder j continues to have $z_j(b) = b$ while z_i solves (FOC′) with $z_j' = 1$ and $z_j(b) = b$.

Finally, I observe that while this proposed algorithm leads to continuous and weakly increasing bidding functions, they are not strictly increasing because of the discontinuities in z_i. Hence, for asymmetric bidders, the definition of a regular equilibrium would need to be relaxed to allow for weakly increasing bids.

5.2 Simpler auctions

In the symmetric independent and regular case, I could have used a very simple auction to accomplish my stated goal: Each bidder submits a bid b_i and price r_i, but instead of being rewarded with revenue only when losing the auction, bidder i receives a payment of $r_i \vee b_{-i}^{(2)}$ if $b_{-i}^{(1)} \geq r_i$. In effect, the seller "simulates" the revenue that the bidder would receive from setting a reserve price of r_i. In the symmetric independent and regular case with distribution $F(v)$, the optimal reserve price is independent of the number of bidders and simply solves

$$1 - F(v) - vf(v) = 0.$$

so bidders will report a reserve price solving this first-order condition. The seller can then implement this reserve price for the remaining bidders.

This auction generates no incentives to shade to throw the auction, since the simulated revenue is received regardless of whether the bidder wins the good. However, with irregular, asymmetric or correlated distributions, there is no simple formula for the optimal reserve price, nor an easy way to relate it to some ex-ante reserve price that does not condition on whether or not a bidder is the loser. Indeed, for a modified version of the example from Section 4.3 depicted in Figure 4, each bidder would suggest a price of 0.21 in the simulated revenue auction, even though 0.88 is the optimal anonymous reserve with two bidders.

5.3 Uniqueness

In Section 4.1, I characterized two necessary conditions for a symmetric regular equilibrium, namely that when the bid function is differentiable, (FOC) must be satisfied, and $\beta(\overline{v}) = \frac{\overline{v}}{1+\widehat{\alpha}}$. I strongly suspect that the equilibrium of Algorithm 1 is unique among the class of regular equilibria, though I have not proven this result. Other authors have investigated uniqueness of auction equilibria in similar settings, notably Lizzeri and Persico (2000) and Lebrun (2006). Lizzeri and Persico (2000) in particular use a notion of regularity that is analogous to my own, though my requirement that bidders bid their values when indifferent is unnecessary for the auctions they consider. This assumption could be dispensed with by modifying the auction format by adding a small probability event that bidder i is sold the good at a randomly drawn price r if $r \leq b$, where r is drawn from

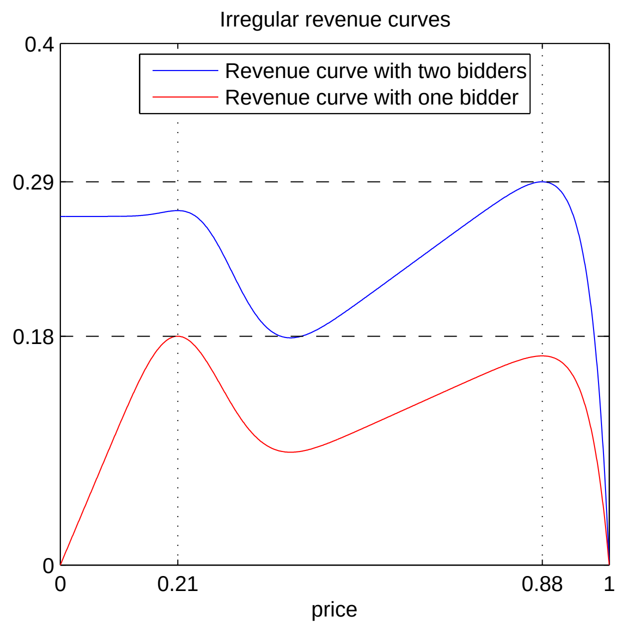

Figure 4: Comparison of revenue with one bidder versus two, when values are independently drawn from a $B[\alpha = 11, \beta = 30]$ distribution with probability 0.81 and from a $B[\alpha = 25, \beta = 1]$ distribution with probability 0.19. In the simulated revenue auction, each bidder suggests a price of 0.21, even though 0.88 is the optimal anonymous reserve price with two bidders.

the cumulative distribution $G(r)$ with support equal to $\mathbb{R}_+$. This extra incentive to bid close to one's value interacts smoothly with (FOC), and does not substantively change the structure of equilibrium. I hope to consider the question of uniqueness in a future version of this manuscript.

5.4 Extension to general type spaces

At the heart of my arguments is that bidders should not shade too much in equilibrium, because of the requirement that the sum of marginal surplus and marginal revenue must be zero if bidders shade a positive amount. I showed that that for the RSA, in order for the bidding function to solve (FOC), bids must be at least $\frac{v}{1+\widehat{\alpha}}$. However, in order to prove that the bound holds, I had to construct an equilibrium, which required the monotonicity property.

In more general type spaces, there is an easy way to achieve a similar bound using a first-order condition. Consider an auction in which the bidder elicits bids and prices, as in the RSA. With probability $1 - \alpha$, the seller picks a bidder to consult at random and uses that bidder's suggested reserve price, also as in the RSA. With probability α, the seller simply uses the second-highest bid as the price for the winner. Crucially, the share of revenue that goes to the consulted bidder is α^2 times realized revenue.

Let us consider the marginal incentive to shade using this auction format. If the pricing constraint is not binding, then there is no incentive to shade, and bidding one's value is a weakly undominated strategy. If the pricing constraint binds, then the marginal surplus is

$$\begin{aligned}\frac{\partial S(v,\beta(w))}{\partial w} &= \left[(1-\alpha)(v-\rho(w))\mathbb{I}_{\beta(w)\geq\rho(w)} + \alpha(v-\beta(w))\right] \\ &\quad \cdot F_{v_{-ij}^{(1)}|v_i,v_j}(w|v,w)f_{v_j|v_i}(w|v) \\ &\geq \alpha(v-\beta(w))F_{v_{-ij}^{(1)}|v_i,v_j}(w|v,w)f_{v_j|v_i}(w|v)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial R(v,\beta(v))}{\partial w} &= \alpha^2\Bigg[\left(F_{v_j,v_{-ij}^{(1)}|v_i}(\overline{v},w|v) - F_{v_j,v_{-ij}^{(1)}|v_i}(w,w|v)\right)\beta'(w) \\ &\quad - \beta(w)F_{v_{-ij}^{(1)}|v_i,v_j}(w|v,w)f_{v_j|v_i}(w|v)\Bigg] \\ &\geq -\alpha^2\beta(w)F_{v_{-ij}^{(1)}|v_i,v_j}(w|v,w)f_{v_j|v_i}(w|v)\end{aligned}$$

If it is true that

$$\frac{\partial S(v,\beta(w))}{\partial w} + \frac{\partial R(v,\beta(v))}{\partial w} = 0,$$

then

$$\begin{aligned}\alpha^2\beta(w) &\geq \alpha(v-\beta(w)) \\ \implies \beta(w) &\geq \frac{v}{1+\alpha}.\end{aligned}$$

Thus, if an equilibrium exists for this more general mechanism, and if bidders bid their values unless they have a strict incentive to shade (as they would be if the trick referred to in the Section 5.3 were used), then $\beta(w)$ must be at least $\frac{v}{1+\alpha}$. As a result, bounds similar to those of Proposition 2 would obtain. However, existence is no small order, as has been pointed out in the literature (see Reny, 1999; Athey, 2001; Reny and Zamir, 2004).

6 Conclusion

This paper has considered a setting in which the buyers know the distribution of values, and therefore know the optimal reserve price, but the seller does not. The seller wishes to have the bidders communicate enough of what they know so that the seller can obtain the the greater revenue associated with a well-chosen reserve price, but the seller also desires that the bidders communicate as little information in as concise a manner as possible. This leads us to a mechanism in which each bidder simply recommends a reserve price for the seller to use in the event that the bidder loses the auction. Truthful reporting of the reserve price is incentivized with revenue sharing.

This rule distorts bidders incentives, and to some extent pushes down the equilibrium bid distribution relative to the value distribution. Nonetheless, the distortions are small when the seller only shares a small amount of revenue, and the seller is able to extract virtually all of the revenue that he would obtain if he knew the distribution and set the optimal anonymous reserve price. In that sense, this mechanism accomplishes the seller's goal.

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A Proofs

Proof of Lemma 2. A type v can win for sure at price $\beta(\overline{v})$ and obtain a payoff of

$$\mathbb{E}\left[v - \rho(v_j) \vee \beta\left(v_{-ij}^{(1)}\right) \middle| v_i = v\right].$$

On the other hand, by bidding $\beta(v)$, a bidder with valuation v can obtain

$$\mathbb{E}\left[\left(v - \rho(v_j) \vee \beta\left(v_{-ij}^{(1)}\right)\right)\mathbb{I}_{\rho(v_j)\vee\beta(v_{-ij}^{(1)})<\beta(v)} \middle| v_i = v\right] \\ + \widehat{\alpha}\,\mathbb{E}\left[\rho(v) \vee \beta\left(v_{-i}^{(2)}\right)\mathbb{I}_{\beta(v_{-i}^{(1)})>\rho(v)} \middle| v_i = v\right].$$

The difference is

$$\mathbb{E}\left[\left(v - \rho(v_j) \vee \beta\left(v_{-ij}^{(1)}\right)\right)\mathbb{I}_{\rho(v_j)\vee\beta(v_{-ij}^{(1)})>\beta(v)} - \widehat{\alpha}\rho(v) \vee \beta\left(v_{-i}^{(2)}\right)\mathbb{I}_{\beta(v_{-i}^{(1)})>\rho(v)} \middle| v_i = v\right] \\ > \mathbb{E}\left[\left(v - \rho(v_j) \vee \beta\left(v_{-ij}^{(1)}\right) - \widehat{\alpha}\rho(v) \vee \beta\left(v_{-i}^{(2)}\right)\right)\mathbb{I}_{\beta(v_{-i}^{(1)})>\beta(v)} \middle| v_i = v\right],$$

since $\rho(v_j) \geq \beta(v_j)$ and $\rho(v) \geq \beta(v)$. Clearly, $\rho(v) \leq \beta(\overline{v})$, since otherwise no revenue would be generated. Hence, this quantity is at least

$$\mathbb{E}\left[\left(v - \beta(\overline{v})(1+\widehat{\alpha})\right)\mathbb{I}_{v_{-i}^{(1)}>v} \middle| v_i = v\right].$$

If $\beta(\overline{v}) < \frac{\overline{v}}{1+\overline{\alpha}}$, then this quantity is positive for v sufficiently close to $\overline{v}$, in which case deviating to $\beta(\overline{v})$ will be attractive for such a v.

On the other side, if $\beta(\overline{v}) > \frac{\overline{v}}{1+\overline{\alpha}}$, then type $\overline{v}$'s payoff from bidding $\beta(\overline{v})$ is

$$\mathbb{E}\left[\overline{v} - \rho(v_j) \vee \beta(v_{-ij}^{(1)}) \middle| v_i = \overline{v}\right],$$

whereas the payoff from bidding $\rho(v) < \beta(\overline{v})$ and setting the same price is

$$\mathbb{E}\left[\left(\overline{v} - \rho(v_j) \vee \beta\left(v_{-ij}^{(1)}\right)\right)\mathbb{I}_{\rho(v_j)\vee\beta(v_{-ij}^{(1)})\leq\rho(v)} \middle| v_i = \overline{v}\right] + \widehat{\alpha}\mathbb{E}\left[\rho(v) \vee \beta\left(v_{-i}^{(2)}\right)\mathbb{I}_{\beta(v_{-i}^{(1)})\geq\rho(v)} \middle| v_i = \overline{v}\right],$$

so that the difference is

$$\mathbb{E}\left[\left(\overline{v} - \rho(v_j) \vee \beta\left(v_{-ij}^{(1)}\right)\right)\mathbb{I}_{\rho(v_j)\vee\beta\left(v_{-ij}^{(1)}\right)\geq\rho(v)} \middle| v_i = \overline{v}\right] - \widehat{\alpha}\mathbb{E}\left[\rho(v) \vee \beta\left(v_{-i}^{(2)}\right)\mathbb{I}_{\beta(v_{-i}^{(1)})\geq\rho(v)} \middle| v_i = \overline{v}\right].$$

Since $\rho(v_j) \geq \beta(v_j)$, I conclude

$$\rho(v_j) \vee \beta\left(v_{-ij}^{(1)}\right)\mathbb{I}_{\rho(v_j)\vee\beta\left(v_{-ij}^{(1)}\right)\geq\rho(v)} \geq \rho(v)\mathbb{I}_{\rho(v_j)\vee\beta\left(v_{-ij}^{(1)}\right)\geq\rho(v)},$$

and also $\mathbb{I}_{\rho(v_j)\vee\beta\left(v^{(1)}_{-ij}\right)\geq\rho(v)} \leq \mathbb{I}_{\beta\left(v^{(1)}_{-i}\right)\geq\rho(v)}$. Hence, the difference is at most

$$\mathbb{E}\left[\left(\overline{v}-\rho\left(v\right)\left(1+\widehat{\alpha}\right)\right)\mathbb{I}_{\beta\left(v^{(1)}_{-i}\right)\geq\rho(v)}\middle| v_i=\overline{v}\right],$$

which must be negative for $\rho(v)$ close to $\beta(\overline{v})$, since $\rho(v)$ is being squeezed to $\beta(\overline{v}) > \frac{\overline{v}}{1+\widehat{\alpha}}$. □

Proof of Proposition 1. Consider a point $\underline{w}^k$, which is the supremum of $v < \overline{w}^k$ such that $\beta_k(v) > v$. By continuity, it must be that $\beta_k(\underline{w}^k) = \underline{w}^k$, so the derivative β' at such a point is

$$\beta_k'(\underline{w}^k) = \underline{w}^k g(\underline{w}^k, \underline{w}^k|\underline{w}^k)$$

If $\beta_k'(\underline{w}^k) > 1$, then $\beta_k(v) \geq v$ for $v \in [\underline{w}^k, \underline{w}^k+\epsilon)$, which contradicts the definition of $\underline{w}^k$. Hence, it must be that $g(\underline{w}^k, \underline{w}^k|\underline{w}^k) \leq \frac{1}{\underline{w}^k}$.

Next, I show that $g(\overline{w}^k, \overline{w}^k|\overline{w}^k) \geq \frac{1}{\overline{w}^k}$, as long as $\underline{v} < \underline{w}^k < \overline{w}^k$. Clearly this is true at $\overline{w}^0 = \overline{v}$, since $g(\overline{w}^k, \overline{w}^k|\overline{w}^k)$ blows up at that point. For $k > 0$, according to the constructed equilibrium, $\beta(\overline{w}^k) = \overline{w}^k$ and $\overline{w}^k \in r^*(\overline{w}^k, \overline{w}^k)$. This inclusion follows from upper-hemicontinuity of r^*. Moreover, on $(\overline{w}^k-\epsilon, \overline{w}^k]$ it must be that $\beta(v) = v$. As a result, marginal revenue at the reserve price r is exactly (2). Clearly, marginal revenue is positive if $g(\overline{w}^k, \overline{w}^k|\overline{w}^k) < \frac{1}{\overline{w}^k}$, which contradicts $\underline{w}^k \in r^*(\underline{w}^k, \underline{w}^k)$.

To summarize, it must be that $g(\overline{w}^k, \overline{w}^k|\overline{w}^k) \geq \frac{1}{\overline{w}^k}$ and $g(\underline{w}^k, \underline{w}^k|\underline{w}^k) \leq \frac{1}{\underline{w}^k}$. By **A2**, there can be at most finitely many points at which $g(v, v|v)$ changes sign. Finally, $\underline{w}^k$ cannot coincide with $\overline{w}^{k+1}$, since if $R(v, v) > R(v, \underline{w}^k)$ for v near $\underline{w}^k$, marginal revenue must be negative so that $1 - v\, g(v, v|v) < 0$ for v near $\underline{w}^k$ (since this function has finitely many zeros), so

$$\beta'(v) \leq \frac{\beta(v)(1+\widehat{\alpha})-v}{\widehat{\alpha}v} < 1,$$

since $\beta(v) \leq v$, so $\beta_k(v) < v$ for v in $(\underline{w}^k-\epsilon, \underline{w}^k]$. This contradicts the definition of $\underline{w}^k$. Hence, any sequence of decreasing v^k for which $g(v^k, v^k|v^k)$ alternates sign (weakly) must terminate after finitely many steps. □

Proof of Lemma 3. This is obviously true on regions $[\overline{w}^k, \underline{w}^{k-1}]$, when $\beta(v) = v$. Second, suppose that $\beta(v) < \frac{v}{1+\widehat{\alpha}}$ for some $v \in [\underline{w}^k, \overline{w}^k]$. Since $\beta(v)$ is continuous and $\beta(\overline{w}^k) \geq \frac{\overline{w}^k}{1+\widehat{\alpha}}$, the following quantity is well defined:

$$\widetilde{v} = \inf\left\{w \geq v \middle| \beta(w) \geq \frac{w}{1+\widehat{\alpha}}\right\}.$$

Then $\beta(w) < \frac{w}{1+\widehat{\alpha}}$ for all $w \in (v, \widetilde{v})$. By the mean value theorem, there exists $\widehat{w} \in (v, \widetilde{v})$ such that

$$\beta'(\widehat{w}) = \frac{\frac{\widetilde{v}}{1+\widehat{\alpha}} - \beta(v)}{\widetilde{v}-v} > \frac{1}{1+\widehat{\alpha}} > 0.$$

But by (FOC), $\beta'(\widehat{w}) > 0$, a contradiction. $\square$

Proof of Lemma 4. The derivative of this function is

$$\begin{aligned}
\frac{dU(v,\beta(w),\beta(w))}{dw} &= \left(\frac{\partial S(v,b)}{\partial b}\bigg|_{b=\beta(w)} + \frac{\partial R(v,r)}{\partial r}\bigg|_{r=\beta(w)} \right) \beta'(w) \\
&= (n-1)(v-\beta(w)) F_{v_{-ij}^{(1)}|v_i,v_j}(w|v,w) f_{v_j|v_i}(w|v) \\
&\quad + \alpha \Bigg[\beta'(w) \left(F_{v_j,v_{-ij}^{(1)}|v_i}(\overline{v},w|v) - F_{v_j,v_{-ij}^{(1)}|v_i}(w,w|v) \right) \\
&\quad - \beta(w) F_{v_{-ij}^{(1)}|v_i,v_j}(w|v,w) f_{v_j|v_i}(w|v) \Bigg] \\
&= (n-1) \left(F_{v_j,v_{-ij}^{(1)}|v_i}(\overline{v},w|v) - F_{v_j,v_{-ij}^{(1)}|v_i}(w,w|v) \right) \\
&\quad \cdot \Big[(v - \beta(w)(1+\widehat{\alpha})) g(w,w|v) + \widehat{\alpha} \beta'(w) \Big].
\end{aligned}$$

Substituting in (FOC) yields

$$\begin{aligned}
\frac{dU(v,\beta(w),\beta(w))}{dw} &= (n-1) \left(F_{v_j,v_{-ij}^{(1)}|v_i}(\overline{v},w|v) - F_{v_j,v_{-ij}^{(1)}|v_i}(w,w|v) \right) \\
&\quad \cdot \Big[(\beta(w)(1+\widehat{\alpha}) - w) g(w,w|w) - (\beta(w)(1+\widehat{\alpha}) - v) g(w,w|v) \Big].
\end{aligned}$$

Note that $\beta(w)(1+\widehat{\alpha}) \geq w$ by Lemma 3, so the first term is positive.

Take $w < v$. Then $g(w,w|w) \geq g(w,w|v)$. Hence, it must be that

$$\begin{aligned}
&(\beta(w)(1+\widehat{\alpha}) - w) g(w,w|w) - (\beta(w)(1+\widehat{\alpha}) - v) g(w,w|v) \\
&\geq (\beta(w)(1+\widehat{\alpha}) - w) g(w,w|v) - (\beta(w)(1+\widehat{\alpha}) - v) g(w,w|v) \\
&= (v-w) g(w,w|w) \geq 0.
\end{aligned}$$

If $w > v$, then $g(w,w|w) < g(w,w|v)$, and

$$\begin{aligned}
&(\beta(w)(1+\widehat{\alpha}) - w) g(w,w|w) - (\beta(w)(1+\widehat{\alpha}) - v) g(w,w|v) \\
&\leq (\beta(w)(1+\widehat{\alpha}) - w) g(w,w|v) - (\beta(w)(1+\widehat{\alpha}) - v) g(w,w|v) \\
&= (v-w) g(w,w|v) \leq 0.
\end{aligned}$$

$\square$

Proof of Lemma 5. To prove the first part of the Lemma, note that

$$\log \left(F_{v_j,v_{-ij}^{(1)}|v_i}(\overline{v},r|v) - F_{v_j,v_{-ij}^{(1)}|v_i}(r,r|v) \right) = -\int_{x=\underline{v}}^{\overline{v}} g(x,r|v) dx,$$

which is increasing in v. Hence,

$$F_{v_j,v_{-ij}^{(1)}|v_i}(\overline{v},r|v) - F_{v_j,v_{-ij}^{(1)}|v_i}(r,r|v)$$

is also increasing in v, as is $1 - rg(r,r|v)$. Given the expression for $\frac{\partial R(v,r)}{\partial r}$, this proves the claim.

By definition, $R(\overline{w}^k, \overline{w}^k) \geq R(\overline{w}^k, r)$ for all $w \in \underline{W}^k$. Hence, if $v < \overline{w}^k$, the difference is

$$\begin{aligned} R(v,w) - R(v,\overline{w}^k) &= \int_{x=\overline{w}^k}^{w} \frac{\partial R(v,r)}{\partial r} \\ &\leq \int_{x=\overline{w}^k}^{w} \frac{\partial R(\overline{w}^k,r)}{\partial r} \\ &= R(\overline{w}^k,w) - R(\overline{w}^k,\overline{w}^k) \leq 0. \end{aligned}$$

The other direction is significantly more complicated. Our goal is to show that the integral

$$R(v,\underline{w}^{k-1}) - R(v,w) = \int_{x=w}^{\underline{w}^{k-1}} \frac{\partial R(v,x)}{\partial x} dx$$

is non-negative for $v \geq \underline{w}^{k-1}$. First, I will show that $r^*(v) = \inf(r^*(v,v))$ is monotonically increasing on $\underline{W}^k$. Take $v > v'$ and $x \in r^*(v,v)$. Then $R(v,x) \geq R(v,y)$ for all $y \geq v$. This implies that $R(v',x) \geq R(v,y)$ for all $y \geq x$, by the fact that $\frac{\partial R(v,r)}{\partial r}$ is increasing in v. Hence, if $x \notin r^*(v',v')$, it means that there must be a $y \in r^*(v',v')$ such that $R(v',y) > R(v',x)$ and hence $y < x$. Thus, either (1) $r^*(v) \in r^*(v',v')$, in which case weakly increasing is obvious, or (2) $r^*(v) \notin r^*(v',v')$, in which case there must exist $y < r^*(v)$ in $r^*(v',v')$.

With this monotonicity result in hand, for any point in the image of $x = r^*(v)$ on $(\underline{w}^k, \overline{w}^{k-1}]$, x must be an optimal price for type v, and moreover must satisfy an interior first-order condition. Otherwise, if the constraint $x \geq v$ were binding, it would be the case $R(v,v) > R(v,w)$ for all $w \in (v, \underline{w}^{k-1}]$, which contradicts the definition of $\overline{w}^k$. Hence,

$$\left.\frac{\partial R(v,x)}{\partial x}\right|_{x=r^*(v)} = 0 \leq \left.\frac{\partial R(v,x)}{\partial x}\right|_{x=r^*(v)}.$$

In other words, for any x in the image of r^*, marginal revenue is non-negative for type $\underline{w}^{k-1}$. On the other hand, for any $x \in [\overline{w}^k, \underline{w}^{k-1}] \setminus r^*([\overline{w}^k, \underline{w}^{k-1}])$, it must be that x is passed over at a jump discontinuity of the monotonic function r^*. Let K be the countable collection of intervals that result from jump discontinuities, i.e., the set of $[a,b]$ such that $a = r^*(v')$ and $b = \lim_{v'' \downarrow w} r^*(v'')$ with $b > a$. Clearly, $r^*([\overline{w}^k, \underline{w}^{k-1}]) \cup \{I \in K\} = [\overline{w}^k, \underline{w}^{k-1}]$.

For each $[a,b] \in K$ which is the jump at v', it must be that $\{a,b\} \subset r^*(v',v')$, for if $R(v',a) > R(v',b)$, then this will also be true for $v'' > v'$ but nearby. Moreover, it must be that $R(v',a) \geq$

$R(v', w)$ for any $w \in [a, b]$. Hence,

$$0 = \int_{x=w}^{b} \frac{\partial R(v, x)}{\partial x} dx \leq \int_{x=w}^{b} \frac{\partial R(\underline{w}^{k-1}, x)}{\partial x} dx.$$

Finally, this shows that

$$\int_{x=w}^{\underline{w}^k} \frac{\partial R(v, x)}{\partial x} dx = \int_{(\cup_{I \in K} I) \cap [w, \underline{w}^{k-1}]} \frac{\partial R(v, x)}{\partial x} dx + \int_{r^*([w, \underline{w}^{k-1}])} \frac{\partial R(v, x)}{\partial x} dx \geq 0,$$

which proves the other direction. It is straightforward to repeat the argument with $r^*(v)$ instead of $\underline{w}^k$. For that case, I would show that

$$\int_{x=w}^{\underline{w}^k} \frac{\partial R(v, x)}{\partial x} dx \geq 0,$$

which is established by analogous arguments.

For the second part of the Lemma, marginal revenue has the same sign as

$$\begin{aligned}
\frac{dU(v, \beta(w))}{dw} &= \alpha \left(F_{v_j, v_{-ij}^{(1)} | v_i}(\overline{v}, w|v) - F_{v_j, v_{-ij}^{(1)} | v_i}(w, w|v) \right) \left[\beta'(w) - \beta(w) g(w, w|v) \right] \\
&= \alpha \left(F_{v_j, v_{-ij}^{(1)} | v_i}(\overline{v}, w|v) - F_{v_j, v_{-ij}^{(1)} | v_i}(w, w|v) \right) \\
&\quad \cdot \left[\frac{\beta(w) - w}{\widehat{\alpha}} g(w, w|w) + \beta(w)(g(w, w|w) - g(w, w|v)) \right].
\end{aligned}$$

Clearly $\beta(w) \leq w$, so the first term in the brackets is non-positive. Also, for $w \geq v$, **A1** implies that $g(w, w|v) \geq g(w, w|w)$, so the second term is non-positive as well. Hence, marginal revenue is non-positive at $w \in \overline{W}^k$ if $w > v$. □

Proof of Theorem 1. To start, fix a valuation v and consider deviations to some (b, r). If $r < v$, then it is without loss of generality to consider the deviation (r, r), since $S(v, b)$ is weakly increasing as long as $b \leq v$. On the other hand, if $r > v$, I can consider deviations of the form (v, r), since $S(v, b)$ is weakly decreasing for $b \geq v$. These are referred to as downward and upward deviations, respectively.

Downward deviations. I prove a base step and an inductive step. The base step considers the cases where $v \in \overline{W}^k$ or $v \in \underline{W}^k$.

For $v \in \overline{W}^k$, consider deviations to some $r \leq \beta(\overline{w}^k)$, since either (1) $\overline{w}^k = \overline{v}$, and it would never be profitable to set a price above the support of bids, or (2) $\beta(\overline{w}^k) = \overline{w}^k > v$. Since β is continuous, and deviations not in the support of bids would never be attractive, the deviation r is equal to $\beta(w)$ for some w. Lemma 4 shows that $\beta(v)$, $\beta(v)$ is weakly better than any downward deviation $\beta(w)$ for $w \in \overline{W}^k$.

Now suppose that $v \in \underline{W}^k$. Lemma 5 shows that $R(v, \underline{w}^k) \geq R(v, r)$ for all $r \in \underline{W}^k$. Also, any downward deviation would entail lower surplus as well, since $S(v, b)$ is weakly increasing for $b \leq v$.

As a result, there can be no profitable downward deviations to $r = \beta(r) \in \underline{W}^k$. This concludes the base step.

For the first half of the inductive step, suppose that deviating to $(\beta(\overline{w}^k), \beta(\overline{w}^k))$ is not profitable, where $\overline{w}^k \leq v$. Again, Lemma 4 shows that $U(v, \beta(w), \beta(w))$ is weakly decreasing for $w \in \overline{W}^k$, so any deviation to $\beta(\overline{W}^k)$ is weakly worse than $\beta(\overline{w}^k)$.

For the second half, suppose that deviating to $(\underline{w}^k, \underline{w}^k)$ is not profitable. Lemma 5 again shows that $R(v, \underline{w}^k) \geq R(v, r)$ for all $r \in \underline{W}^k$, and $S(v, b)$ is weakly decreasing, so there are no profitable deviations in $\underline{W}^k$.

Hence, for any $\overline{W}^k$ with $\overline{w}^k < v$ or $\underline{W}^k$ with $\underline{w}^k < v$, it cannot be that there are any profitable downward deviations to $\beta(w)$ with $w \in \overline{W}^k$ or $w \in \underline{W}^k$.

Upward deviations. I again show a base step and an inductive step, as in the downward case. If $v \in \overline{W}^k$, the downward case has shown that (v, v) is not a profitable deviation. By Lemma 5, marginal revenue is non-positive if $w > v$, so there cannot be profitable deviations to some $(v, \beta(w))$ with $w \in \overline{W}^k$ and $\beta(w) \geq v$, since this implies $w \geq v$ as well.

On the other hand, if $v \in \underline{W}^K$, Lemma 5 shows that $R(v, \underline{w}^k) \geq R(v, r)$ for all $r \in \underline{W}^k$. This concludes the base step for upward deviations.

I have already shown that marginal revenue is non-positive on $\overline{W}^k$. Hence, the deviation $(v, \underline{w}^k)$ is weakly better than $(v, \beta(w))$ for $w \in \overline{W}^k$.

For the other half of the inductive step, Lemma 5 shows that $R(v, \overline{w}^{k+1}) \geq R(v, r)$ for all $r \in \underline{W}^k$. So if $(v, \overline{w}^{k+1})$ is not profitable, then neither is such a deviation (v, r). This concludes the inductive step. $\square$