Optimal auction design with common values: An informationally-robust approach*

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Abstract

A profit-maximizing Seller has a single unit of a good to sell. The bidders have a pure common value that is drawn from a distribution that is commonly known. The Seller does not know the bidders' beliefs about the value and thinks that beliefs are designed adversarially by Nature to minimize profit. We construct a *strong maxmin solution* to this joint mechanism design and information design problem, consisting of a mechanism, an information structure, and an equilibrium, such that neither the Seller nor Nature can move profit in their respective preferred directions, even if the deviator can select the new equilibrium. The mechanism and information structure solve a family of maxmin mechanism design and minmax information design problems, regardless of how an equilibrium is selected. The maxmin mechanism takes the form of a *proportional auction*: each bidder submits a one-dimensional bid, the aggregate allocation and aggregate payment depend on the aggregate bid, and individual allocations and payments are proportional to bids. We report a number of additional properties of the maxmin mechanisms, including what happens as the number of bidders grows large and robustness with respect to the prior over the value.

Keywords: Mechanism design, information design, optimal auctions, profit maximization, common value, information structure, maxmin, Bayes correlated equilibrium, direct mechanism.

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1 Introduction

1.1 Background and motivation

We study the design of profit-maximizing mechanisms when the bidders have a pure common value for the good being sold, but partial and differential information about that value. Potential applications include the sale of natural resources or financial assets, where to a first order all bidders have the same preferences over the market value of the resource or the future cash flows of the asset.

Although common-value auctions have been studied since the early days of auction theory, relatively little is known about optimal common-value auctions. When bidders' signals are independent and one-dimensional, Bulow and Klemperer (1996) have argued that a variation of the English auction is optimal under a condition that signals associated with higher expected values are not too precise. In the perhaps more natural case where signals are correlated through the common value, such as in the mineral rights model, McAfee, McMillan, and Reny (1989) and McAfee and Reny (1992) construct mechanisms that extract virtually all of the surplus by having the bidders bet on other bidders' information. While the full-surplus extracting mechanisms are theoretically interesting, there are a number of reasons why they may not be practically useful, including that the designer may not know exactly how information is correlated, and the optimal mechanism may be too complicated for bidders to use.

This discussion points to some conceptual challenges in optimal auction design. First, optimal mechanisms vary widely with the model of bidders' information, e.g., whether and how signals are correlated. At the same time, it is hard to determine, either through measurement or introspection, which model of information is empirically relevant, and hence which of the potentially optimal mechanisms is appropriate. Moreover, relatively little is known about how optimal mechanisms behave if the model is misspecified, which raises the question of whether Bayesian optimal mechanisms should be used in the presence of such model uncertainty. Note that these problems also arise in non-common value mechanism design. When values are private, these concerns are partially allayed by the broad consensus that the independent private value model is a useful benchmark. In contrast, there is no comparably canonical model when values are common.

To address these issues, we model a Seller who knows the distribution of the common value, but regards bidders' beliefs and higher-order beliefs about the value as ambiguous. These beliefs are modeled as a common-prior *information structure*. The Seller is concerned about model misspecification, and believes that the information structure is chosen adversarially by Nature to minimize equilibrium profit.

1.2 Main results

This joint mechanism design and information design problem is not a standard zero-sum game, as a given mechanism and information structure need not have a unique equilibrium. What is the resulting profit level when there are multiple equilibria, or, for that matter, if no equilibrium exists? We address the issues of equilibrium multiplicity and existence by employing a new solution concept: A *strong maxmin solution* is a triple of a mechanism, an

information structure, and an equilibrium strategy profile, with the property that neither the Seller nor Nature can move equilibrium profit in their respective preferred directions by changing the mechanism or information structure, respectively, even if the deviator can select the equilibrium. In other words, the mechanism and equilibrium maximize profit holding the information structure fixed, and the information structure and equilibrium minimize profit holding the mechanism fixed, and these statements remain true regardless of how an equilibrium is selected. This solution can be interpreted as Nash equilibrium of the game in which the Seller chooses mechanisms and an adversarial Nature chooses information structure and a particular equilibrium is played, and it remains a Nash equilibrium regardless of which equilibrium is selected. The solution has an associated *profit guarantee*, which is both a tight lower bound on equilibrium profit for the mechanism across all information structures, and a tight upper bound on equilibrium profit for the information structure across all mechanisms.

Our main result (Theorem 1) is the construction of a strong maxmin solution. The maxmin mechanism is what we term a *proportional auction*: Bidders submit one-dimensional bids. The aggregate allocation (the total probability the good is allocated) and the aggregate transfer (the sum of the bidders' payments) depend only on the aggregate bid. Individual allocations and transfers are proportional to bids.[1] In benchmark cases, the aggregate allocation is linearly increasing in the aggregate bid until it hits one and is constant thereafter. An interpretation is that bids are "demands" for a quantity of the good, which are completely filled when the aggregate demand is less than the available supply, and otherwise the good is rationed proportionally. Bidders pay a constant price per-unit that depends only on the aggregate bid.

In the minmax information structure, bidders' signals are i.i.d. draws from the standard exponential distribution, and the expectation of the value given the signals depends only on the aggregate signal. Finally, in equilibrium, each bidder submits a bid that is equal to their signal.

When the number of bidders is large, the profit guarantee is approximately the entire ex ante gains from trade, i.e., the expectation of the value under the prior minus the cost of production (or zero if the expected value is less than the cost). The guarantee also seems to be a substantial share of surplus even when the number of bidders is small. For example, when there are two bidders and the value is standard uniform and production is costless, the maxmin proportional auction guarantees the Seller at least 56 percent of the expected value as profit.

Before presenting our main theorem, we give a heuristic derivation of the solution. First, the minmax information structure is constructed so that the Seller is indifferent between a wide range of mechanisms. In particular, the Seller is indifferent between *all* mechanisms with the same aggregate allocation, and whenever the optimal mechanism rations the good, the Seller is indifferent between allocating and not allocating.

[1] Proportional auctions can be seen as a generalization of the Tullock contest, which corresponds to the case where the good is always allocated and the aggregate transfer is linear in the aggregate bid. The closest real-world auction that we can find are the "voucher auctions" used to privatize Soviet state assets in the 1990s (Krishna, 2009, p. 184).

The mechanism is then constructed to be a profit maximizing direct mechanism on the minmax information structure with the additional property that the optimal profit at the minmax information is minimum equilibrium profit across all information structures. Importantly, messages in the maxmin mechanism are "normalized" to be signals in the minmax information structure. We refer to this as the *double revelation principle*: The maxmin mechanism is a profit-maximizing direct mechanism on the minmax information structure, and the minmax information structure is a profit-minimizing correlated equilibrium on the maxmin mechanism. The existence of a solution of this form is a non-trivial result, and it does not follow from the standard revelation arguments.

The requirement that profit be minimized at the minmax information structure reduces to a pair of differential equations involving the mechanism's allocation and transfer rules. The first equation pins down the divergence of the allocation rule, which we refer to as the *aggregate allocation sensitivity*. The second differential equation, which we refer to as *profit-incentive alignment*, links ex post profit to the bidders' local incentives. In particular, it pins down the difference between the divergence of the transfers and the aggregate transfer, which we term the *aggregate excess growth*. The proportional auction solves these two equations and also satisfies the revelation constraints at the minmax information structure.

In the definition of a strong maxmin solution, the profit guarantee is only compared with profit in other equilibria, where we hold fixed the mechanism and vary the information structure, or vice versa. It is possible that there are alternative strong maxmin solutions with different profit guarantees, but this can only happen if either the Seller or Nature cannot deviate to the objects we construct because it leads to a game with no equilibria. A distinct but related concern is that minor modifications to the maxmin mechanism, such as discretizing or bounding the message space, would lead to qualitatively new equilibria that dramatically change the profit guarantee.

We address these questions of uniqueness and robustness of the strong maxmin solution in the following manner. A solution is *finitely approximable* if there are finite mechanisms and finite information structures with associated profit guarantees that are arbitrarily close to the solution's profit guarantee. We show that our solution is finitely approximable (Theorem 3). Moreover, any finitely approximable solution must have the same value (Theorem 4). In fact, the approximating finite mechanisms are simply discrete proportional auctions, and the finite information structures correspond to finite partitions of signals in the minmax information structure. If we restrict the Seller and Nature to finite mechanisms and information structures, respectively, then these approximations attain the sup-inf and inf-sup of profit, regardless of the equilibrium selection rule (Corollary 1). Thus, the strong maxmin solution we construct is a limit of ϵ-equilibria of the zero-sum game in which the Seller and Nature choose finite mechanisms and finite information structures, respectively.

As a last topic, we consider the behavior of maxmin proportional auctions as the number of bidders grows large and the value distribution and cost are held fixed. In the many-bidder limit, the optimal profit guarantee converges to the ex ante gains from trade. This generalizes an analogous result of Du (2018) when there is common knowledge of gains from trade. Moreover, this limit obtains even if the good is always sold, and at the same optimal rate of $O(1/\sqrt{N})$. Finally, we show that the profit guarantee converges to the ex ante gains from trade *even if the prior is misspecified*.

The maxmin modeling approach allows us to identify new mechanisms with desirable theoretical properties that hold uniformly across information structures and equilibria. There is a conceptual tension, however, between the extreme ambiguity aversion of the Seller and the common knowledge of the information structure among the agents. In particular, why does the Seller not simply ask the agents to report the information structure? In our view, the information structure and Bayes Nash equilibrium are an as-if description of behavior, which we hope is a reasonable approximation. We do not want to interpret these objects literally as something that either the Seller or the bidders could fully articulate. The maxmin mechanism does not require the bidders to report higher order beliefs, nor does the Seller need to specify a model of beliefs in order to compute the maxmin mechanism. In that sense, it is consistent with real-world limitations on knowledge and communication.[2] That being said, the assumption of large ambiguity is as extreme as the assumption that the Seller knows the information structure exactly. We view it as a benchmark and a starting point for future work on informationally-robust optimal mechanisms. We return to this point in the conclusion of the paper.

1.3 Related literature

This paper lies at the intersection of the literatures on mechanism design and information design. We build on the seminal paper of Myerson (1981) on optimal auction design, and also subsequent work by Bulow and Klemperer (1996). We also draw heavily from the literature on robust predictions (Bergemann and Morris, 2013, 2016).

The most closely related papers are Du (2018) and Bergemann, Brooks, and Morris (2016). Du (2018) solves our maxmin mechanism design problem in the limit as the number of bidders goes to infinity and the production cost is zero. Specifically, Du constructs a sequence of mechanisms and associated lower bounds on profit that converge to the expected surplus in the many-bidder limit. The mechanisms from Du (2018) do not achieve the optimal profit guarantee when the number of bidders is finite and more than one.[3] In contrast, Bergemann, Brooks, and Morris (2016) construct what is essentially a strong maxmin solution for the special case of two bidders and two possible values. Interestingly, the minmax information structure they identify coincides with the one we construct, but the maxmin mechanisms are different. We discuss this further in Section 5.

Chung and Ely (2007), Yamashita (2016), and Chen and Li (2018) also study maxmin mechanism design when the Seller does not know the information structure but when values are private and when the Seller preferred equilibrium is selected. In contrast, we focus on a common value environment. Other conceptually related studies of robust mechanism design are Neeman (2003), Brooks (2013), Yamashita (2015), Carroll (2017), Bergemann, Brooks, and Morris (2019), and the literature on algorithmic mechanism design (e.g., Hartline and Roughgarden, 2009).

[2] To be sure, some features of the information structure are relatively easy to express, such as first order expectations. But we are skeptical as to whether real-world bidders or auction designers can describe the fine details of higher order beliefs.

[3] Du (2018) also solves the present problem in the case of one bidder. With one bidder and binary values, our model reduces to that of Carrasco et al. (2018).

The rest of the paper proceeds as follows. Section 2 describes our model and solution concept. Section 3 informally derives the strong maxmin solution. Section 4 presents the main result. Section 5 discusses uniqueness of the profit guarantee. Section 6 explores the many-bidder limit. Section 7 is a conclusion.

2 Model

2.1 Primitives

A Seller has a unit of a good that can be sold to one of $N \geq 2$ bidders. The bidders have a pure common value for the good v which is distributed according to the cumulative distribution function H on $\mathbb{R}_+ = [0, \infty)$. The support of H, denoted V, is assumed to be bounded, with $\underline{v}$ and $\overline{v}$ denoting the minimum and maximum, respectively. We also assume that $\underline{v} < \overline{v}$.[4]

Bidders' preferences over probabilities of receiving the good, q_i, and the amount they pay for it, t_i, are represented by the state-dependent utility index $vq_i - t_i$.

The good costs $c \geq 0$ to produce. The Seller's profit from the profiles of allocations $q = (q_1, \ldots, q_N)$ and transfers $t = (t_1, \ldots, t_N)$ is $\sum_{i=1}^{N}(t_i - cq_i)$. We assume that the expected value is strictly larger than c.[5]

For technical reasons, we assume that the left tail of H is not too thin. To state the precise condition, we need the following definition: For a cumulative distribution F on $\mathbb{R}$, the associated *quantile function* is

$$F^{-1}(\alpha) = \min\{x \mid F(x) \geq \alpha\}$$

Because F is increasing and right-continuous, the set of values with cumulative probability higher than α is closed, so this minimum is well-defined. Now, let G_N denote the distribution of the sum of N independent draws from the exponential distribution with unit arrival rate, also known as an Erlang distribution, which is a special case of the Gamma distribution. (This object features prominently in our analysis and is given explicitly in equation (15) below.) The first part of the left-tail assumption is that there exist $\epsilon > 0$ and $\varphi > 1$ such that, for all $x \in [0, \epsilon]$,

$$H^{-1}(G_N(x)) - \underline{v} \leq x^{\varphi}.$$

The second part of the left-tail assumption is that if $\underline{v} > c$, then there exists an $\epsilon > 0$ such that, for all $x', x \in [0, \epsilon]$ such that $x' < x$,

$$\frac{H^{-1}(G_N(x)) - c}{H^{-1}(G_N(x')) - c} \leq \exp(x - x').$$

[4] If not, then the value is common knowledge, and the Seller can easily extract all the surplus.

[5] Otherwise, a trivial solution is that bidders have no information and the good is not sold.

These assumptions are satisfied if there is a mass point at $\underline{v}$ (which implies $H^{-1}(G_N(x)) = \underline{v}$ for x sufficiently small) or if H has a density that is bounded away from zero around $\underline{v}$.[6]

2.2 Information

Fix cumulative distributions F_1 and F_2. Recall that F_1 is a *mean-preserving spread* of F_2 if there exist a probability space and random variables X_1 and X_2 such that X_1 has distribution F_1, X_2 has distribution F_2, and $\mathbb{E}[X_1|X_2] = X_2$. Equivalently, for all $x \in \mathbb{R}$,

$$\int_{y=-\infty}^{x} (F_1(y) - F_2(y))dy \geq 0, \tag{1}$$

and this holds with equality when $x = \infty$ (Blackwell and Girshick, 1954; Rothschild and Stiglitz, 1970).

An *information structure* $\mathcal{S}$ consists of (i) a measurable set S_i of signals for each bidder i, (ii) a joint distribution $\pi \in \Delta(S)$ where $S = \times_{i=1}^{N} S_i$, and (iii) a function $w : S \to \mathbb{R}$ such that H is a mean-preserving spread of the distribution of $w(s)$. For a profile of signals s, $w(s)$ represents the interim expectation of v conditional on s.[7]

2.3 Mechanisms

A *mechanism* $\mathcal{M}$ consists of measurable sets of messages M_i for each i and measurable mappings $q_i : M \to [0, 1]$ and $t_i : M \to \mathbb{R}$ for each i, where $M = \times_{i=1}^{N} M_i$ is the set of message profiles, such that $\sum_{i=1}^{N} q_i(m) \leq 1$. For technical reasons, we assume that t_i is bounded below (although it may be negative).

We further restrict attention to mechanisms that satisfy a condition we call *participation security*: For every i, there exists $0 \in M_i$ such that $v\, q_i(0, m_{-i}) - t_i(0, m_{-i}) \geq 0$ for every $v \in V$ and every $m_{-i} \in M_{-i}$. By sending this message, bidder i is assured a non-negative payoff ex post, no matter what messages are sent by the other bidders.

2.4 Equilibrium

A mechanism $\mathcal{M}$ and an information structure $\mathcal{S}$ comprise a game of incomplete information. A *(behavioral) strategy* for bidder i is a transition kernel $\beta_i : S_i \to \Delta(M_i)$. A profile of strategies $\beta = (\beta_1, \ldots, \beta_N)$ is identified with a transition kernel that associates to each $s \in S$ the product measure $\beta_1(s_1) \times \cdots \times \beta_N(s_N)$ on $\Delta(M)$.

[6] If $\underline{v} > c$ and H has a density $h(v) \geq b > 0$ for $v \in [\underline{v}, \underline{v} + \epsilon']$, then

$$\frac{dH^{-1}(G_N(x))}{dx} = \frac{g_N(x)}{h(H^{-1}(G_N(x)))} \leq \frac{g_N(x)}{b}, \quad \frac{d\log(H^{-1}(G_N(x)) - c)}{dx} = \frac{g_N(x)}{(H^{-1}(G_N(x)) - c) \cdot h(H^{-1}(G_N(x)))} \leq \frac{g_N(x)}{(\underline{v} - c)b}$$

whenever $G_N(x) \leq H(\underline{v} + \epsilon')$, where g_N is the density for G_N. As is evident from the formula for g_N in (14) below, $g_N(x) \to 0$ at a rate of x (or faster) as $x \to 0$. Thus, there exists an $\epsilon > 0$ such that for every $x \in [0, \epsilon]$, $H^{-1}(G_N(x)) - \underline{v} \leq x^{\varphi}$ for $\varphi \in (1, 2)$ and $d\left(\log(H^{-1}(G_N(x)) - c)\right)/dx \leq 1$. This implies the left-tail assumption.

[7] This definition is equivalent to one in which we specify the joint distribution of the signals and the value. Since the interim expectation is the key object in our analysis, this formulation simplifies notation.

Given a strategy profile β, bidder i's payoff is

$$U_i(\mathcal{M}, \mathcal{S}, \beta) = \int_S \int_M (w(s)q_i(m) - t_i(m))\beta(dm|s)\pi(ds).$$

Note that since w, q, and $-t$ are all bounded above, this integral is always well-defined. A strategy profile β is a *(Bayes Nash) equilibrium* if for all i and strategies β_i',

$$U_i\left(\mathcal{M}, \mathcal{S}, \beta\right) \geq U_i\left(\mathcal{M}, \mathcal{S}, (\beta_i', \beta_{-i})\right).$$

The set of equilibria is denoted by $B(\mathcal{M}, \mathcal{S})$. Expected profit is

$$\Pi(\mathcal{M}, \mathcal{S}, \beta) = \int_S \int_M \sum_{i=1}^{N} (t_i(m) - cq_i(m))\beta(dm|s)\pi(ds).$$

2.5 Solution concept

We will shortly introduce the solution concept employed in this paper. This solution concept is motivated by the following simultaneous-move game between Seller and Nature: Fix a measurable set X. Define $\mathbf{M}(X)$ to be the set of participation-secure mechanisms in which each bidder's message space is of the form $M_i \cup \{1, \ldots, k_i\}$ for some measurable $M_i \subseteq X$ and some non-negative integer k_i. Similarly define $\mathbf{S}(X)$ to be the set of information structures where signal spaces are of the form $S_i \cup \{1, \ldots, k_i\}$ for some measurable $S_i \subseteq X$ and some non-negative integer k_i.[8] Let $\mathcal{B}(X)$ denote the set of all selections from the equilibrium correspondence B on the subset of $\mathbf{M}(X) \times \mathbf{S}(X)$ for which an equilibrium exists. Given a selection $\beta^* \in \mathcal{B}(X)$, we define the game $\mathcal{G}(X, \beta^*)$ where Seller and Nature simultaneously choose actions in $\mathbf{M}(X)$ and $\mathbf{S}(X)$. The Seller's payoff is $\Pi(\mathcal{M}, \mathcal{S}, \beta^*(\mathcal{M}, \mathcal{S}))$ and Nature's payoff is $-\Pi(\mathcal{M}, \mathcal{S}, \beta^*(\mathcal{M}, \mathcal{S}))$ if $B(\mathcal{M}, \mathcal{S}) \neq \emptyset$, and both parties' payoffs are minus infinity if $B(\mathcal{M}, \mathcal{S}) = \emptyset$.

By fixing the equilibrium selection, we have formulated the joint mechanism design and information design problem as a standard non-cooperative game. A Nash equilibrium of this game $(\mathcal{M}, \mathcal{S})$ is non-trivial if a bidder equilbrium exists for the game $(\mathcal{M}, \mathcal{S})$. Such an equilibrium can be understood as a pair of an informationally-robust mechanism and a worst-case informational environment, which rationalize one another as optimal, given the equilibrium selection rule.[9] A concern with this modeling approach is that whether $(\mathcal{M}, \mathcal{S})$ is a Nash equilibrium may depend on the particular equilibrium selection rule. This motivates us to consider pairs $(\mathcal{M}, \mathcal{S})$ which are non-trivial Nash equilibria for all selections β^*.[10] As Proposition 1 below shows, this notion of a "selection-invariant" non-

[8] The finitely many extra messages and signals allows us to add messages to direct revelation mechanisms in order to make them participation secure. This construction is used in the proof of Proposition 1.

[9] By assuming that the payoff from bidder-equilibrium non-existence is minus infinity, we implicitly restrict the Seller to only consider mechanisms for which an equilibrium exists on $\mathcal{S}$, and correspondingly for Nature. We can view this as capturing a belief of the Seller that the information structure $\mathcal{S}$ is possible, and that they must ensure the mechanism is well behaved in that environment.

[10] Many mechanisms with multiple equilibria can be perturbed to select a particular equilibrium, with a negligible effect on profit. For this reason, it is not surprising that Nash equilibrium payoffs in $\mathcal{G}(X, \beta^*)$ are invariant to β^*. Nonetheless, the equilibrium selection rule could have a significant impact on which mechanisms and information structures are part of Nash equilibria.

trivial Nash equilibrium is equivalent to the following solution concept: A *strong maxmin solution* consists of a triple $(\mathcal{M}, \mathcal{S}, \beta)$ of a mechanism, an information structure, and a strategy profile, with profit $\Pi = \Pi(\mathcal{M}, \mathcal{S}, \beta)$, such that the following are satisfied:

1. For any mechanism $\mathcal{M}'$ and any equilibrium β' of $(\mathcal{M}', \mathcal{S})$, $\Pi \geq \Pi(\mathcal{M}', \mathcal{S}, \beta')$;
2. For any information structure $\mathcal{S}'$ and any equilibrium β' of $(\mathcal{M}, \mathcal{S}')$, $\Pi \leq \Pi(\mathcal{M}, \mathcal{S}', \beta')$;
3. β is an equilibrium of $(\mathcal{M}, \mathcal{S})$.

We refer to Π as the *profit guarantee* of the solution.[11]

Conditions 1 and 2 say that the Seller and Nature cannot improve their payoff by deviating, even if the deviator selects the equilibrium. Condition 3 says that the profit guarantee is not vacuous, and there exists an equilibrium at which Π is attained. In fact, the definition implies that for a strong maxmin solution $(\mathcal{M}, \mathcal{S}, \beta)$, all equilibria of $(\mathcal{M}, \mathcal{S})$ must generate profit Π.

The following result connects the strong maxmin solution to Nash equilibria of the previously defined non-cooperativce game:

Proposition 1. *Fix a pair $(\mathcal{M}, \mathcal{S})$. Then the following are equivalent:*

(i) There exist strategies β such that $(\mathcal{M}, \mathcal{S}, \beta)$ is a strong maxmin solution.

(ii) There exist an X such that $(\mathcal{M}, \mathcal{S})$ is a non-trivial Nash equilibrium of $\mathcal{G}(X, \beta^*)$ for all $\beta^* \in \mathcal{B}(X)$.

The proof is a straightforward application of the revelation principles for mechanism design and information design, and is relegated to the Appendix.

The main result of our paper is the construction of a strong maxmin solution. Proposition 1 shows that we can equivalently interpret this solution as a Nash equilibrium of the game between Seller and Nature, regardless of how we select an equilibrium.[12] How we should model equilibrium selection depends on a number of considerations. On the one hand, β^* could select the profit-minimizing equilibrium if the Seller is concerned for robustness with respect to equilibrium selection. On the other hand, the literature on partial implementation typically assumes that the mechanism designer can use their prominence to coordinate bidders on the designer's preferred equilibrium. It is therefore normatively appealing that the solution we construct does not depend on additional assumptions about equilibrium selection.

[11] This definition nominally depends on qualifiers over *all* mechanisms and information structures, which are of course not well-defined. However, it is clearly without loss to restrict attention to the set of incentive compatible and participation-secure direct mechanisms on $\mathcal{S}$ and truthful equilibria in condition 1 and to restrict attention to Bayes correlated equilibria on $\mathcal{M}$ and obedient strategies in condition 2.

[12] This invariance criterion is reminiscent of Govindan and Wilson (2009), who look at sequential equilibrium outcome that is invariant to the extensive form representation of the underlying reduced normal form.

3 A roadmap to the solution

We rigorously construct a strong maxmin solution at the beginning of Section 4, and Theorem 1 verifies that the construction is indeed a solution. This section gives an informal derivation. To be clear, our purpose is to develop intuition, and the proof of Theorem 1 does not depend on the present discussion.

3.1 The structure of the solution

The strong maxmin solution we construct is denoted $(\overline{\mathcal{M}}, \overline{\mathcal{S}}, \overline{\beta})$. The high level structure is as follows. Signals in the information structure and messages in the mechanism are elements of $\overline{M}_i = \overline{S}_i = [0, \infty)$. In addition, the equilibrium strategies specify that each bidder send a message that is equal to their signal: for all i and s_i, $\overline{\beta}_i(s_i) = s_i$. Thus, a common language is used for signals and messages. One interpretation is that the maxmin mechanism $\overline{\mathcal{M}}$ is a *direct mechanism* on the minmax information structure $\overline{\mathcal{S}}$, whereby a message is a "report" of which signal a bidder received, and bidders report truthfully in equilibrium. An equally valid interpretation is that $\overline{\mathcal{S}}$ is a *Bayes correlated equilibrium* (BCE) on $\overline{\mathcal{M}}$, whereby a signal is a "recommendation" of a message to send, and in equilibrium, bidders obey their recommendations.

If we held the information structure fixed and maximized profit across mechanisms and equilibria, then the well-known revelation principle (Myerson, 1981) says that it is without loss of generality to restrict attention to direct mechanisms. Similarly, if the mechanism were fixed and we minimized profit across information structures and equilibria, then it is without loss to restrict attention to BCE, which is a kind of revelation principle for games (Bergemann and Morris, 2013, 2016). In the present model, both the mechanism and the information structure are endogenous, so the standard revelation arguments do not apply.[13] It is therefore a result that there exists a solution that admits the same normalization. We refer to this as the *double revelation principle*.

3.2 The minmax information structure

We next describe the rest of the minmax information structure $\overline{\mathcal{S}}$, from which we subsequently derive the maxmin mechanism. First, signals in $\overline{\mathcal{S}}$ turn out to be independently distributed. This is intuitive, for if signals were correlated, the Seller could extract surplus by having bidders make bets about others' beliefs, similar to the full-surplus extracting mechanisms mentioned in the introduction (although it may not be possible to extract the entire surplus).

Given independence, the rest of the form of $\overline{\mathcal{S}}$ can be understood using the celebrated *revenue-equivalence formula* of Myerson (1981), suitably adapted to the common value setting. Let us suppose that the marginal distribution of each signal s_i admits a density f_i.

[13] Holding information fixed, any mechanism $\mathcal{M}$ and equilibrium β have an equivalent direct mechanism $\mathcal{M}'$ in which truth telling is an equilibrium. But $\mathcal{M}'$ may have other equilibria with no counterpart in $\mathcal{M}$, and our solution concept considers how profit varies across *all* equilibria. Similarly, replacing a given information structure and equilibrium with the corresponding direct information may lead to a qualitatively different set of equilibria.

Revenue equivalence says that expected profit is, up to a constant, the expectation of the *virtual value* of the bidder who receives the good. When the value function is differentiable, the virtual value of bidder i when the signal profile is s is[14,15]

$$\psi_i(s) = w(s) - c - \frac{1 - F_i(s_i)}{f_i(s_i)} \frac{\partial w(s)}{\partial s_i},$$

where F_i is the cumulative distribution of bidder i's signal. Thus, the virtual value is equal to the gains from trade minus an information rent. The latter is the product of the inverse hazard rate, which is the relative measure of types who receive an information rent when s_i is allocated the good, and $\partial w(s)/\partial s_i$, which quantifies the value of bidder i's private information.

Among independent-signal information structures, it is without loss of generality to normalize the signals to be exponential with a unitary arrival rate:[16] $F_i(x) = 1 - \exp(-x)$. As a result, the inverse hazard rate is constant and equal to one, and drops out of the virtual value formula.

The remaining degree of freedom is the value function $w(s)$. To develop intuition for the minmax value function, we may ask, which value function would be worst for the Seller? Drawing on experience from zero-sum games, we might suspect that the worst case would be associated with indifference between lots of mechanisms. This would roughly mean that $\overline{\mathcal{S}}$ is hard to respond to, in that while lots of mechanisms perform reasonably well, no mechanism stands out as exceptional.

In fact, there is a class of value functions that make the Seller indifferent as to who is allocated the good for *every* signal profile, namely those of the form $w(s) = w(\Sigma s)$, where $\Sigma s = s_1 + \cdots + s_N$ is the *aggregate signal*.[17] (We maintain this convention for the sum of a vector's elements throughout the paper.) As a result, the interim expected value is equally sensitive to all signals, and all bidders have the same virtual value of $w(\Sigma s) - c - w'(\Sigma s)$.

We are still free to choose the particular function of the aggregate signal. An important variant of our model, discussed in Section 4.3, is the *must-sell case*, where the good has to be sold with probability one. This is in contrast to the general *can-keep case*, where the Seller can withhold the good. Note that Σs has the Erlang cumulative distribution G_N introduced in Section 2, and g_N denotes the associated density.[18] All bidders have the same virtual value, so profit is

$$\int_{x=0}^{\infty} (w(x) - c - w'(x)) g_N(x) dx. \tag{2}$$

[14]In the classic formulation of Myerson (1981), bidder i's virtual value is their value minus the inverse hazard rate. We obtain this formula if there are bidder-specific values $w_i(s)$ and signals are normalized so that $w_i(s) = s_i$, in which case the partial derivative is identically one. The formula reported here is a special case of one that appears in Bulow and Klemperer (1996).

[15]Our formal arguments in Section 4 sidestep the direct calculation of virtual values, to avoid technical complications associated with whether there is an integral representation for the bidders' indirect utilities.

[16]Given any w and $F_1, \ldots, F_N$, an equivalent information structure would be one with i.i.d. exponential signals and the value function $w(F_1^{-1}(1 - \exp(-s_1)), \ldots, F_N^{-1}(1 - \exp(-s_N)))$.

[17]We hope we do not create confusion by using the same notation for the interim value as a function of the signal profile and as a function of the aggregate signal.

[18]Both G_N and g_N have closed-form expressions, given as equations (14) and (15) below.

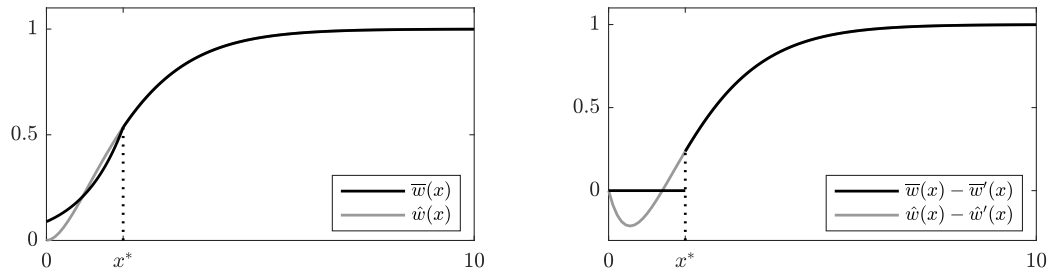

Figure 1: Value functions and virtual value functions when $N = 2$, $v \sim U[0, 1]$, and $c = 0$.

This formula assumes that transfers are set so that the bidder with the lowest signal receives a payoff of zero, which maximizes revenue subject to local incentive compatibility and participation security (since the latter implies that interim payoffs are non-negative). Note that the expectation of $w(x)$ must equal the ex ante expected value under H. Thus, to minimize profit, the value function should maximize the expected slope. This is achieved by the *fully-revealing value function* $\widehat{w}(x) = H^{-1}(G_N(x))$, where H^{-1} is the quantile function for H. This value function matches aggregate signals and values comonotonically, so that their percentiles are perfectly correlated. It is fully revealing in that there is no uncertainty about the value, conditional on the join of the bidders' information. It is intuitive that $\widehat{w}$ minimizes profit, since it maximizes the bidders' private information about the value. Figure 1 illustrates the fully-revealing value and virtual value functions when $N = 2$, $c = 0$, and v is standard uniform, so that $\widehat{w}(x) = G_2(x)$.

For some value distributions, $\widehat{w}$ is also the minmax value function when the Seller can keep the good. This is not the case for the uniform distribution. The right-hand panel of Figure 1 shows that the virtual value is strictly negative when the aggregate signal is low, so that the Seller strictly prefers to withhold the good. The Seller can be made strictly worse off by adding noise to the bidders' information so that the Seller is indifferent between selling and not selling. This requires that the virtual value is zero, i.e., $w(x) - c - w'(x) = 0$. Equivalently, the *gains function* $\gamma(x) = w(x) - c$ (for interim expected gains from trade) is of the form $k \exp(x)$ for some $k \in \mathbb{R}_+$.

In the uniform example, we can replace the fully-revealing gains function $\widehat{\gamma}(x) = \widehat{w}(x) - c$ on an interval $[0, x^*]$ with an exponential segment, to obtain

$$\overline{\gamma}(x) = \begin{cases} \overline{\gamma}(0) \exp(x) & \text{if } x \leq x^*; \\ \widehat{\gamma}(x) & \text{if } x > x^*. \end{cases}$$

We choose $\overline{\gamma}(0)$ and x^* so that H remains a mean-preserving spread of the distribution of the interim expected value, and so that the exponential curve connects continuously with the fully-revealing gains function. This is the black curve in Figure 1, which is the minmax gains function when the Seller can keep the good.

More generally, the sign of the fully-revealing virtual value might switch back and forth, and there could be many exponential segments. In Section 4.1, we describe a general procedure that transforms the fully-revealing gains function so that the resulting virtual value is everywhere non-negative. We refer to this as *grading the gains function*, meaning we

decrease the derivative of the gains function so that it does not grow faster than exponential. The graded gains and value functions are denoted by $\overline{\gamma}$ and $\overline{w}$, respectively, and the resulting information structure is $\overline{\mathcal{S}}$. Proposition 2 shows that profit on $\overline{\mathcal{S}}$ is at most

$$\overline{\Pi} = \int_{x=0}^{\infty} \overline{\gamma}(x) g_{N-1}(x) dx. \tag{3}$$

This formula can be obtained from (2) via integration by parts, using the fact that $dg_N(x)/dx = g_{N-1}(x) - g_N(x)$. $\overline{\Pi} + c$ is also the largest posted price at which all bidders would be willing to purchase the good, which is the expectation of the value given a signal $s_i = 0$. Thus, $\overline{\Pi}$ is exactly optimal profit on $\overline{\mathcal{S}}$, and a posted price is an optimal mechanism (although it is not a maxmin mechanism!)

3.3 Sufficient conditions for an optimal profit guarantee

We now derive a maxmin mechanism $\overline{\mathcal{M}}$ from $\overline{\mathcal{S}}$. At first glance, it seems that we do not learn very much from the requirement that $\overline{\mathcal{M}}$ maximize profit on $\overline{\mathcal{S}}$, because so many mechanisms are optimal. We learn a great deal, however, from the requirements that (i) $\overline{\mathcal{S}}$ and $\overline{\beta}$ minimize equilibrium profit on $\overline{\mathcal{M}}$ and (ii) minimum profit is $\overline{\Pi}$, as we now explain.

Fix a mechanism $\mathcal{M}$ with message space $M_i = \overline{S}_i$ for all i. As reviewed in the introduction, the problem of minimizing profit in $\mathcal{M}$ across information structures and equilibria is equivalent to minimizing profit across BCE.[19] We briefly review this solution concept. An *outcome* of $\mathcal{M}$ is a joint distribution over values and message profiles $\sigma \in \Delta(V \times M)$ such that the marginal distribution of v is H. The associated profit is the expectation of $\Sigma(t - c\,q)$ under σ. A BCE is an outcome that is *obedient*: for all i and m_i, m_i is a best response in M_i to the distribution of (v, m_{-i}) under σ and conditional on m_i. An outcome σ is *consistent with an information structure* $\mathcal{S}$ *and strategies* β if there exists a kernel $K : S \to \Delta(V)$ such that $w(s)$ is the expectation of V under the measure $K(dv|s)$ and

$$\sigma(dv, dm) = \int_S \pi(ds)\beta(dm|s)K(dv|s),$$

i.e., σ is the marginal on values and messages obtained by integrating out signals. Note that if σ is consistent with $\mathcal{S}$ and β, then they have the same expected profit. It is a result of Bergemann and Morris (2013, 2016) that an outcome is a BCE if and only if it is consistent with some $\mathcal{S}$ and equilibrium β.[20] A fortiori, minimum profit across all BCE is equal to minimum profit across all information structures and equilibria.

Let us therefore examine the problem of minimizing profit in $\mathcal{M}$ across BCE. It turns out that the only obedience constraints that are relevant for our problem are those associated

[19] Note that we do not make explicit use of BCE in Section 4 and in the proof of Theorem 1. Nonetheless, these ideas are at work "under the hood."

[20] Strictly speaking, our setup differs from that of Bergemann and Morris in that there are infinitely many states and actions, and we use a different notion of an information structure. The equivalence in our setting can be shown by analogous arguments (although we do not provide such an argument as part of our informal derivation).

with local obedience, i.e., that for all i and m_i,[21]

$$\int_{V\times M_{-i}} \left(v\frac{\partial q_i(m_i, m_{-i})}{\partial m_i} - \frac{\partial t_i(m_i, m_{-i})}{\partial m_i} \right) \sigma(dv, dm_{-i}|m_i) = 0.$$

The problem of minimizing profit subject to the constraint on the marginal distribution of v and local obedience is an infinite dimensional linear program, for which the associated Lagrangian is

$$\begin{aligned}\mathcal{L}(\sigma, \{\alpha_i\}, \lambda) = &\sum_{i=1}^{N} \int_{V\times M} (t_i(m) - cq_i(m))\, \sigma(dv, dm) \\ &+ \sum_{i=1}^{N} \int_{V\times M} \alpha_i(m_i) \left(v\frac{\partial q_i(m)}{\partial m_i} - \frac{\partial t_i(m)}{\partial m_i} \right) \sigma(dv, dm) \\ &+ \int_{V\times M} \lambda(v)\, (H(dv) - \sigma(dv, dm))\,.\end{aligned} \tag{4}$$

This Lagrangian has three terms: profit induced by the BCE, the sum of local obedience constraints times their corresponding multipliers (the functions α_i), and the sum of marginal constraints times their corresponding multipliers (the function λ).

Requirement (i) above is equivalent to saying that an outcome that is consistent with $(\overline{\mathcal{S}}, \overline{\beta})$, denoted by $\overline{\sigma}$, minimizes (4). The key properties of $\overline{\sigma}$ are that messages are i.i.d. standard exponential and $\overline{w}(\Sigma m)$ is the conditional expectation of v given m. A necessary condition for $\overline{\sigma}$ to be the profit-minimizing BCE is that for all (v, m),

$$\sum_{i=1}^{N} \left[t_i(m) - cq_i(m) + \alpha_i(m_i) \left(v\frac{\partial q_i(m)}{\partial m_i} - \frac{\partial t_i(m)}{\partial m_i} \right) \right] - \lambda(v) \geq 0, \tag{5}$$

with the constraint holding as an equality for (v, m) in the support of $\overline{\sigma}$.

We motivated (5) by treating (q, t) as fixed and σ as endogenous. But evaluated at the putative minimizer $\overline{\sigma}$, equation (5) becomes a constraint on the maxmin allocation and transfer rules, involving the as-yet unspecified multipliers λ and $\{\alpha_i\}$.

In fact, the correct multipliers can be deduced from (i) and (ii). Based on the envelope theorem, we can guess that $\lambda(v)$ is the derivative of minimum profit in the maxmin mechanism with respect to the prior probability of v. From (ii), this should coincide with the derivative of $\overline{\Pi}$ with respect to the probability of v, denoted $\overline{\lambda}(v)$. If not, then by making v either more or less likely, we could make minimum profit from the maxmin mechanism increase faster than $\overline{\Pi}$. The function $\overline{\lambda}$ has an explicit formula given in equation (18) below, and we will shortly use the fact that $\overline{\lambda}$ is concave.[22]

As for the multipliers on local obedience, there is an even simpler answer: $\alpha_i(m_i) = 1$ for all i and m_i. This is suggested by the fact that (4) is very similar to the Lagrangian

[21] This is suggested by the fact that only local incentive constraints were used in the revenue equivalence argument that motivated $\overline{\mathcal{S}}$.

[22] For each v, the optimal $\lambda(v)$ must satisfy (5) with equality for some m. As a result, λ is the minimum of a collection of linear functions, indexed by m.

for the linear program of *maximizing* profit given $\overline{\mathcal{S}}$, where we fix $\sigma = \overline{\sigma}$ and optimize over (q_i, t_i), and obedience is reinterpreted as incentive compatibility. As is well known, local incentive constraints bind at the solution, and the optimal multiplier on local incentive compatibility is the inverse hazard rate, which we have normalized to one.

Substituting in these multipliers and letting $Q(m) = \Sigma q(m)$ denote the aggregate allocation, equation (5) reduces to, for all (v, m),

$$\nabla \cdot t(m) - \Sigma t(m) \leq v \nabla \cdot q(m) - \overline{\lambda}(v) - cQ(m), \tag{6}$$

where $\nabla\cdot$ is the divergence operator, and the constraint holds with equality on the support of $\overline{\sigma}$, namely, pairs (v, m) such that $v = \widehat{w}(\Sigma m)$.[23] For a fixed m, the value $v = \widehat{w}(\Sigma m)$ must minimize the right-hand side of (6), and since $\overline{\lambda}$ is concave and hence right-differentiable, it must be that

$$\nabla \cdot q(m) = \overline{\lambda}'(\widehat{w}(\Sigma m)). \tag{7}$$

We refer to the left-hand side of (7) as the *aggregate allocation sensitivity*. In fact, $\overline{\lambda}'(\widehat{w}(x))$ can be computed in closed form, and we denote it by $\overline{\mu}(x)$. When the value function is full-revealing, $\overline{\mu}(x) = (N-1)/x$, and on an interval where the value function is graded, $\overline{\mu}$ is a constant that depends on the end points of the interval. The exact formula is given in equation (17) below.

Substituting (7) into (6), we obtain the following condition on transfers:

$$\nabla \cdot t(m) - \Sigma t(m) = \widehat{w}(\Sigma m)\overline{\mu}(\Sigma m) - \overline{\lambda}(\widehat{w}(\Sigma m)) - cQ(m). \tag{8}$$

The left-hand side of (8) is the *aggregate excess growth*, i.e., the difference between how fast the bidders' transfers grow in their own messages relative to exponential growth. We refer to equation (8) as *profit-incentive alignment*, since it links ex post profit, $\Sigma(t - cq)$, to the bidders' local incentive constraints, $v\nabla \cdot q - \nabla \cdot t$. This ensures that as long as bids are locally optimal, profit cannot fall below $\overline{\Pi}$.

We have been using the profit-minimization program to derive necessary conditions on a maxmin mechanism. But as we argue in Proposition 3, these conditions are actually *sufficient* for a mechanism to guarantee profit of at least $\overline{\Pi}$.[24] Specifically, if a mechanism is such that the aggregate allocation sensitivity is $\overline{\mu}$ and the aggregate excess growth and the aggregate allocation satisfy (8), then profit is at least $\overline{\Pi}$ in all information structures and all equilibria. The proof is essentially an application of the weak duality.

3.4 Construction of a maxmin mechanism

The last step is to construct a mechanism that satisfies (7) and (8) and such that truth telling is an equilibrium at $\overline{\mathcal{S}}$. Note that the latter condition is logically separate from the profit lower bound of Proposition 3.

[23] Recall that $\overline{w}(\Sigma m)$ is just the conditional expectation of v. It may be that $\overline{w}(\Sigma m)$ is not even in the support of H. It is always the case, however, that $\widehat{w}(\Sigma m)$ is in the support of the conditional distribution of v given Σm.

[24] This is true as long as local incentive compatibility (25) holds in any equilibrium. See Lemma 8 below.

Let us start with the allocation. Consider the case with two bidders. In the must-sell case we have $Q(m) = 1$, so $q_2(m_1, m_2) = 1 - q_1(m_1, m_2)$, and the aggregate allocation sensitivity reduces to

$$\frac{\partial q_1(m_1, m_2)}{\partial m_1} - \frac{\partial q_1(m_1, m_2)}{\partial m_2} = \frac{1}{m_1 + m_2}. \tag{9}$$

Now consider a level curve where $m_1 + m_2 = x$. We can then view the left-hand side of (9) as the total derivative of q_1 with respect to m_1 along the parametric curve $m_2(m_1) = x - m_1$, so that integrating both sides, we obtain $q_1(m_1, m_2) = m_1/x + C(x)$ for some function $C(\cdot)$. In order to have $q_1 \in [0, 1]$, we must have $C(x) = 0$, so the allocation probability is simply the bidder's share of the aggregate bid.

More generally, equation (7) is satisfied by the following *proportional allocation rule*:

$$\overline{q}_i(m) = \begin{cases} \frac{1}{N}\overline{Q}(0) & \text{if } \Sigma m = 0; \\ \frac{m_i}{\Sigma m}\overline{Q}(\Sigma m) & \text{if } \Sigma m > 0, \end{cases} \tag{10}$$

where the aggregate allocation $\overline{Q}$ is a function of the aggregate bid and is chosen so that induced aggregate allocation sensitivity is $\overline{\mu}$:

$$\nabla \cdot \overline{q}(m) = \frac{N-1}{\Sigma m}\overline{Q}(\Sigma m) + \overline{Q}'(\Sigma m) = \overline{\mu}(\Sigma m).$$

In equation (16), we give the explicit solution to this differential equation.

This leaves the transfers. Since $\overline{Q}$ has been specified, we can denote by $\overline{\Xi}$ the target aggregate excess growth, which is equal to the right-hand side of (8) and just depends on the aggregate bid. Any solution to (8) must be associated with an apportionment of $\overline{\Xi}$ among the bidders. Indeed, given such an apportionment, it is straightforward to integrate a bidder's excess growth to obtain the implied transfer (cf. the discussion in Footnote 30).

At first glance, there seems to be tremendous flexibility in how we divide the aggregate excess growth. The danger lurking here is that there is no guarantee, for an arbitrary solution of (8), that an equilibrium exists on *any* information structure, let alone $\overline{\mathcal{S}}$. As a result, the profit lower bound implicit in (8) may be vacuous.

It turns out that there is a subtle connection between the incentive compatibility of $\overline{\beta}$ and boundedness of the transfers. In particular, given that the allocation is $\overline{q}$ and that transfers satisfy (8), boundedness implies incentive compatibility, and they are equivalent when $N = 2$. Some obvious solutions to (8), such as equal sharing of the excess growth, result in transfers that sometimes diverge to minus infinity as m_i grows large.[25] We comment further on this connection in the proof of Proposition 4 and Footnote 30.

Thus, the last step to complete the strong maxmin solution is the construction of a bounded transfer rule that satisfies (8). These conditions are satisfied by the *proportional*

[25]Equal sharing means that $\partial t_i(m)/\partial m_i - t_i(m) = \overline{\Xi}(\Sigma m)/N$. Together with the boundary condition $t_i(0, m_{-i}) = 0$, this implies the transfer rule $t_i(m) = \exp(m_i)\int_{x=0}^{m_i} \overline{\Xi}(x + \Sigma m_{-i}) \exp(-x)dx/N$. Since the ex ante expectation of $\overline{\Xi}$ is zero (Lemma 11), it must be that $\int_{x=0}^{\infty} \overline{\Xi}(x + \Sigma m_{-i}) \exp(-x)dx$ is zero on average across m_{-i}. But this integral is non-constant (and generally strictly decreasing in m_{-i}), so sometimes it must be positive and sometimes negative, in which case the transfer tends to $\pm\infty$ as $m_i \to \infty$.

transfer rule:[26]

$$\bar{t}_i(m) = \begin{cases} \frac{1}{N}\overline{T}(0) & \text{if } \Sigma m = 0; \\ \frac{m_i}{\Sigma m}\overline{T}(\Sigma m) & \text{if } \Sigma m > 0, \end{cases} \tag{11}$$

where $\overline{T}(x)$ is the aggregate transfer:

$$\overline{T}(x) = \begin{cases} \underline{v}\overline{Q}(0) & \text{if } x = 0; \\ \frac{1}{g_N(x)} \int_{y=0}^{x} \overline{\Xi}(y) g_N(y) dy & \text{if } x > 0. \end{cases} \tag{12}$$

Note that with this functional form for the transfers, when $\Sigma m > 0$, (8) reduces to

$$\left(\frac{N-1}{x} - 1\right) \overline{T}(x) + \overline{T}'(x) = \overline{\Xi}(x).$$

It is easily verified that (12) satisfies this differential equation, and hence profit-incentive alignment is satisfied.

In addition, boundedness of the transfers is equivalent to boundedness of $\overline{T}(x)$. L'Hôpital's rule, combined with the fact that the ex ante expectation of $\overline{\Xi}$ is zero (Lemma 11), shows that

$$\lim_{x\to\infty} \overline{T}(x) = \lim_{x\to\infty} \frac{\overline{\Xi}(x)}{\frac{N-1}{x} - 1} = -\lim_{x\to\infty} \overline{\Xi}(x). \tag{13}$$

We show below that $\overline{\Xi}$ is bounded as $x \to \infty$, thus verifying that the transfers are also bounded.

Finally, by construction, participation is security is satisfied with $m_i = 0$. Moreover, we show below that $\overline{\Xi}(x)$ is bounded as $x \to 0$ when $\overline{Q}(0) = 0$, and it is approximately $(N-1)/x$ when $\overline{Q}(x) = 1$. Thus, a similar calculation as (13) shows that $\lim_{x\to 0} \overline{T}(x) = \underline{v}\overline{Q}(x)$. As a result, transfers are continuous at 0, and profit-incentive alignment is satisfied everywhere.

We refer to the mechanism comprised of $\overline{q}$ and $\bar{t}$ as a *proportional auction*. A key feature of this mechanism that makes it informationally robust is that it equalizes ex post profit and aggregate ex post local incentives across lots of message profiles. In particular, all message profiles that have the same aggregate message also have the same revenue, cost, and divergences of the allocation and transfer rules, so that they all contribute equally to the weighted sum of profit and local incentives. Thus, just as $\overline{\mathcal{S}}$ induces indifference on the part of the Seller as to how to allocate the good, $\overline{\mathcal{M}}$ induces indifference on the part of Nature, as which message profiles should be played, subject to a given aggregate message.

The optimal aggregate allocation and aggregate transfer functions are plotted for the uniform example in Figure 2. Theorem 1 shows that the proportional auction $\overline{\mathcal{M}}$, the additive-exponential information structure $\overline{\mathcal{S}}$, and the truthful strategies $\overline{\beta}$ together comprise a strong maxmin solution.

[26]This functional form is suggested by the fact that the aggregate excess growth only depends on the aggregate bid, and as we have seen with the allocation rule, the divergence of a proportional rule is similarly only a function of the aggregate bid.

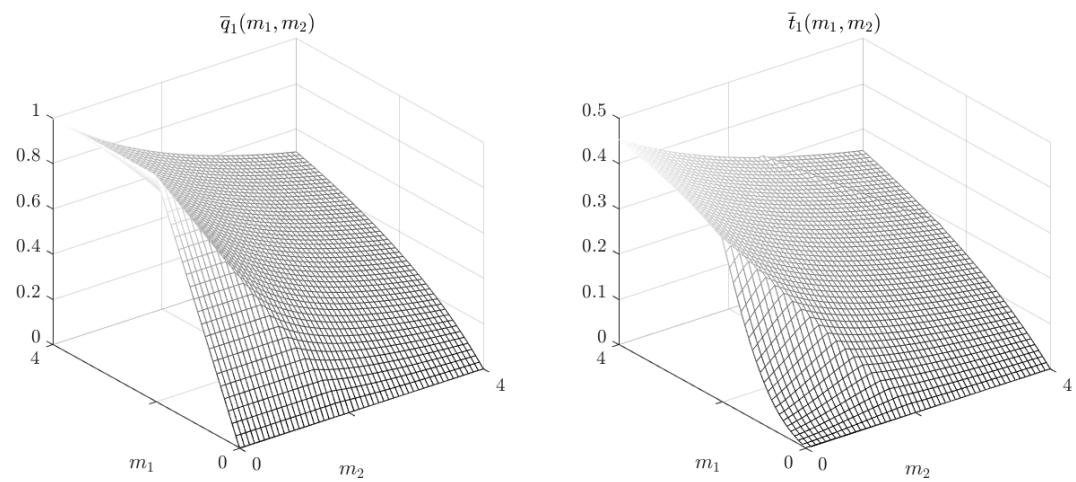

Figure 2: The minmax aggregate allocation and aggregate transfer rules for $m_i \in [0, 5]$, when $N = 2$, $v \sim U[0, 1]$, and $c = 0$.

4 A Strong Maxmin Solution

We now formally construct and characterize a strong maxmin solution. We first completely construct the solution in Section 4.1. We then present our main theorem in Section 4.2, which asserts that constructed triple is indeed a strong maxmin solution. The proof immediately follows. Sections 4.3 and 4.4 discuss two special cases, when the good must be sold and when the value distribution is single crossing, respectively.

4.1 Construction of the solution

4.1.1 Minmax information

The minmax information structure $\overline{\mathcal{S}}$ is defined as follows. The bidders have signal spaces $\overline{S}_i = [0, \infty)$, and the signal distribution is $\overline{\pi}(ds) = \exp(-\Sigma s)ds$, i.e., signals are independent draws from the exponential distribution with arrival rate 1.

The aggregate signal $x = \Sigma s$ has a probability density function

$$g_N(x) = \frac{x^{N-1}}{(N-1)!} \exp(-x) \tag{14}$$

and cumulative distribution function

$$G_N(x) = 1 - \sum_{n=1}^{N} g_n(x). \tag{15}$$

The value function is defined according to the following *grading procedure*. Recall that $\widehat{w}(x) = H^{-1}(G_N(x))$ is the fully-revealing value function, and $\widehat{\gamma}(x) = \widehat{w}(x) - c$ is the fully-revealing gains function. Let

$$\widehat{\Gamma}(x) = \int_{y=0}^{x} \widehat{\gamma}(y) g_N(y) dy.$$

Also let

$$E(x) = \int_{y=0}^{x} \exp(y) g_N(y) dy,$$

which is strictly increasing, and hence it has a continuous inverse E^{-1}. Let $\text{cav}(\widehat{\Gamma} \circ E^{-1})$ denote the smallest concave function that is everywhere above $\widehat{\Gamma} \circ E^{-1}$. We then set $\overline{\Gamma} = \text{cav}(\widehat{\Gamma} \circ E^{-1}) \circ E$, and define

$$\overline{\gamma}(x) = \frac{1}{g_N(x)} \frac{d}{dx} \overline{\Gamma}(x),$$

where the derivative is taken from the right. We further define $\overline{w}(x) = \overline{\gamma}(x) + c$. We refer to $\overline{\gamma}$ and $\overline{w}$ as the *graded gains function* and the *graded value function*, respectively.[27]

4.1.2 Maxmin mechanism

We next construct the maxmin mechanism $\overline{\mathcal{M}}$. The message space is $\overline{M}_i = [0, \infty)$.

We define a *graded interval* to be an interval $[a, b]$ with $a < b$ such that $\overline{\Gamma}(x) = \widehat{\Gamma}(x)$ for $x \in \{a, b\}$ and $\overline{\Gamma}(x) > \widehat{\Gamma}(x)$ for $x \in (a, b)$. As discussed in Section 3, the allocation and transfers are proportional, satisfying (10) and (11). The aggregate allocation function is given by[28]

$$\overline{Q}(x) = \begin{cases} 0 & \text{if } x = 0 \text{ and } [0, b] \text{ is a graded interval for some } b > 0; \\ C(a,b)\frac{x}{N} + D(a,b)\frac{1}{x^{N-1}} & \text{if } x \in [a, b], \text{ where } [a, b] \text{ is a graded interval and } a > 0; \\ 1 & \text{otherwise,} \end{cases} \tag{16}$$

and

$$C(a,b) = N \frac{b^{N-1} - a^{N-1}}{b^N - a^N}, \quad D(a,b) = \frac{b-a}{b^N - a^N} a^{N-1} b^{N-1}.$$

The corresponding aggregate allocation sensitivity is

$$\overline{\mu}(x) = \begin{cases} C(a,b) & \text{if } x \in [a, b), \text{where } [a, b] \text{ is a graded interval,} \\ \frac{N-1}{x} & \text{otherwise.} \end{cases} \tag{17}$$

[27] This procedure is evocative of "ironing" in Myerson (1981) and concavification in Kamenica and Gentzkow (2011). Grading is used to construct the bidders' information that minimizes the Seller's profit, subject to the Seller being always willing to allocate the good and subject to a mean-preserving spread constraint. In Myerson, ironing is used to construct the mechanism that maximizes the Seller's profit, subject to global incentive compatibility. In Kamenica and Gentzkow, concavification is used to construct a receiver's information to maximize a sender's payoff, subject to a mean-preserving spread constraint. We can find no tight link between these problems, beyond the very high-level connection of optimization subject to monotonicity and/or mean-preserving spread constraints.

[28] The functional form of $\overline{Q}$ on a graded interval $[a, b]$ can be derived from the hypotheses that $\overline{Q}(a) = \overline{Q}(b) = 1$ and that the aggregate allocation sensitivity is constant for all message profiles m with $a \leq \Sigma m \leq b$.

The aggregate transfer $\overline{T}$ is defined as follows. First, define

$$\overline{\lambda}(v) = \overline{\Pi} + \int_{x=0}^{\infty} \overline{\mu}(x) G_N(x) d\widehat{w}(x) - \int_{\nu=v}^{\overline{v}} \overline{\mu}(G_N^{-1}(H(\nu))) d\nu, \tag{18}$$

where $\overline{\Pi}$ is the profit guarantee defined in (3), and

$$\overline{\Xi}(x) = \overline{\mu}(x)\widehat{w}(x) - \overline{\lambda}(\widehat{w}(x)) - c\,\overline{Q}(x). \tag{19}$$

The aggregate transfer is then given by (12).

4.1.3 Strategies

Finally, let $\overline{\beta}_i$ be the truthful strategy in the mechanism $\overline{\mathcal{M}}$ under the information structure $\overline{\mathcal{S}}$: For all i and s_i, $\overline{\beta}_i(s_i) = s_i$. This completes the construction of the solution.

4.1.4 Illustration

Various objects in the construction are illustrated in Figure 3 for an example in which $N = 2$, the value is uniformly distributed on $[0, 0.95] \cup [3.95, 4]$, and $c = 0.2$.

The top row depicts the construction of the gains function: From left to right are the gains functions, integrated gains functions, and rescaled gains functions. The fully-revealing versions are in light-gray, and the graded versions are in black. There are two graded intervals, which are denoted $[0, x_1]$ and $[x_2, x_3]$.

The middle row shows the aggregate allocation sensitivity, value multiplier, and aggregate excess growth. Again, fully-revealing objects are in gray and graded counterparts are in black (see Section 4.3 for the discussion of the fully-revealing objects).

The bottom row shows the optimal aggregate allocation and transfer functions.

4.2 Main result

The main result of the paper is the following:

Theorem 1 (Existence)**.** *$(\overline{\mathcal{M}}, \overline{\mathcal{S}}, \overline{\beta})$ is a strong maxmin solution with a profit guarantee of $\overline{\Pi}$ defined by* (3)*.*

The theorem follows from Propositions 2–4. Proposition 2 verifies that $\overline{\mathcal{S}}$ is a well-defined information structure for which equilibrium profit is at most $\overline{\Pi}$. Proposition 3 verifies that $\overline{\mathcal{M}}$ is a well-defined mechanism for which equilibrium profit is at least $\overline{\Pi}$. Finally, Proposition 4 verifies that $\overline{\beta}$ is an equilibrium of $(\overline{\mathcal{M}}, \overline{\mathcal{S}})$.

4.2.1 Upper bound on profit for $\overline{\mathcal{S}}$

We first establish condition 1 in the definition of a strong maxmin solution.

Proposition 2. *$\overline{\mathcal{S}}$ is a well-defined information structure. For all mechanisms $\mathcal{M}$ and equilibria β of $(\mathcal{M}, \overline{\mathcal{S}})$, $\Pi(\mathcal{M}, \overline{\mathcal{S}}, \beta) \leq \overline{\Pi}$.*

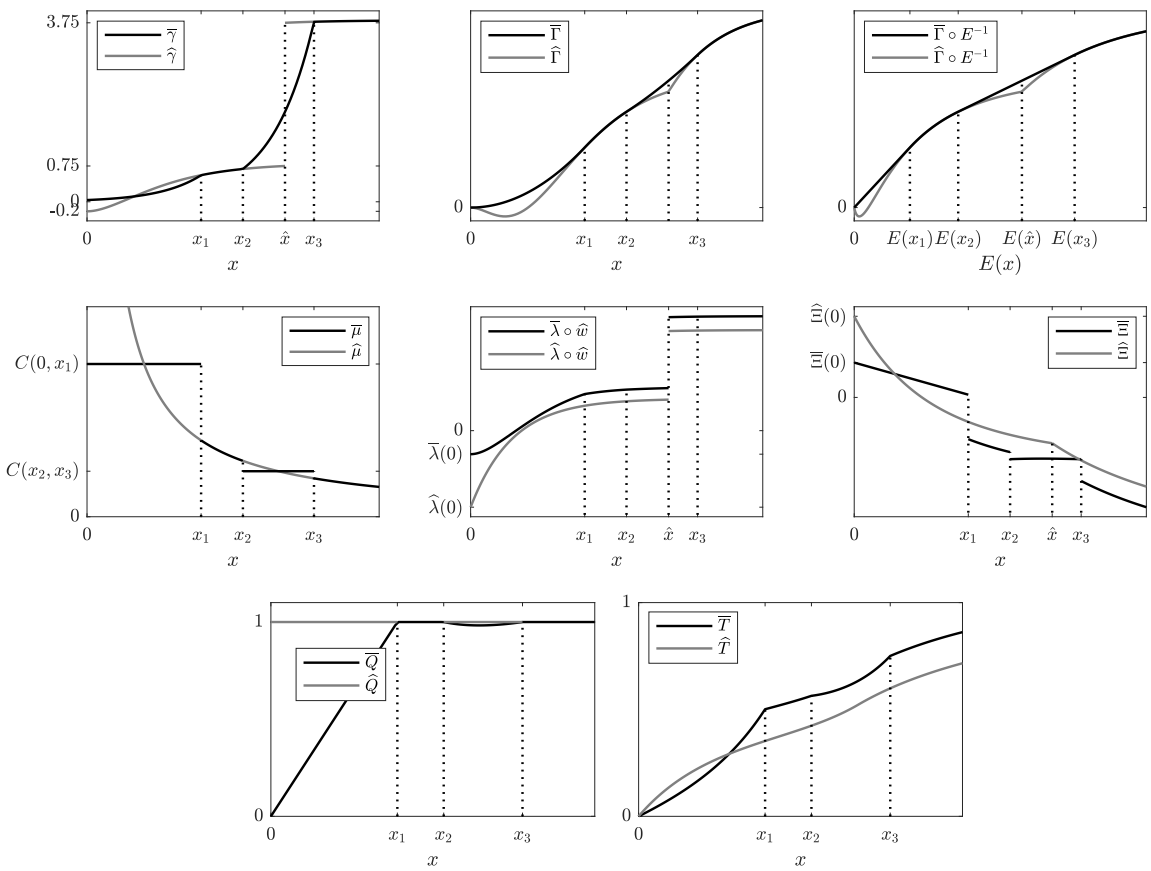

Figure 3: Objects used in the construction of the solution when $N = 2$, a value that is uniform on $[0, 0.95] \cup [3.95, 4]$, and $c = 0.2$. Bars correspond to the can-keep case and hats correspond to the must-sell case discussed in Section 4.3.

We first argue that $\overline{\mathcal{S}}$ is in fact an information structure consistent with the value distribution H.

Lemma 1. *The gains function $\overline{\gamma}$ is a well-defined and increasing function. H is a mean-preserving spread of the distribution of $\overline{w}(\Sigma s)$.*

Proof of Lemma 1. Since $\overline{\Gamma} \circ E^{-1}$ is a concave function, it is continuously differentiable at all but countably many points, and we can extend the derivative by right continuity. Since E is also differentiable, we conclude that $\overline{\Gamma}$ has a right derivative as well. We can therefore define $\overline{\gamma}$ as specified.

We next argue that $\overline{\gamma}(x)$ is continuous. Since $\overline{\Gamma} \circ E^{-1}$ is concave, its right derivative is monotonically decreasing. If the right derivative of $\overline{\Gamma} \circ E^{-1}$ had a downward discontinuity at x, which corresponds to a concave kink in $\overline{\Gamma} \circ E^{-1}$, it would have to occur at a point where $\overline{\Gamma}$ and $\widehat{\Gamma}$ coincide (since $\overline{\Gamma} \circ E^{-1}$ is linear on graded intervals where the two functions differ). This implies that $\widehat{\Gamma} \circ E^{-1}$ also has a concave kink at x, which contradicts the monotonicity of $\widehat{\gamma}$. Thus, we conclude that $\overline{\Gamma} \circ E^{-1}$ has a continuous right derivative, so that $\overline{\gamma}$ is continuous.

We next argue that $\overline{\gamma}$ is increasing. From continuity of $\overline{\gamma}$, it is sufficient to show that it is increasing on graded intervals and on non-graded intervals. If x is such that there is an interval $[x, x+\epsilon)$ on which $\overline{\Gamma}$ coincides with $\widehat{\Gamma}$, then their right-derivatives at x must coincide as well, so that $\overline{\gamma}(x) = \widehat{\gamma}(x)$, where the latter is increasing. In addition, if $[a, b]$ is a graded interval and $x \in [a, b)$, then $\overline{\gamma}$ has an exponential shape, as

$$\frac{d}{dx}\overline{\Gamma}(x) = \frac{d}{dz}(\overline{\Gamma}(E^{-1}(z)))\Big|_{z=E(x)} E'(x) = \frac{\widehat{\Gamma}(b) - \widehat{\Gamma}(a)}{E(b) - E(a)} \exp(x) g_N(x).$$

The (positive) constant is chosen so that $\overline{\Gamma}(x)$ coincides with $\widehat{\Gamma}(x)$ at the end points of the graded interval. Thus, $\overline{\gamma}$ is increasing on graded intervals as well.

We next show that the distribution of $\widehat{\gamma}(\Sigma s)$ is a mean-preserving spread of the distribution of $\overline{\gamma}(\Sigma s)$. The lemma then follows from the observation that the distribution of $\widehat{\gamma}(\Sigma s) + c$ is H, and $\overline{w}(\Sigma s) = \overline{\gamma}(\Sigma s) + c$.

Let $\overline{F}$ and $\widehat{F}$ denote the cumulative distributions of $\overline{\gamma}(x)$ and $\widehat{\gamma}(x)$, respectively, where $x \sim G_N$. Since $\overline{\gamma}$ and $\widehat{\gamma}$ are both increasing, for all $\alpha \in [0, 1]$, $\overline{\gamma}(G_N^{-1}(\alpha)) = \overline{F}^{-1}(\alpha)$ and $\widehat{\gamma}(G_N^{-1}(\alpha)) = \widehat{F}^{-1}(\alpha)$. From the change of variables $y = G_N(x)$, we conclude that

$$\begin{aligned}\int_{y=0}^{\alpha} (\overline{F}^{-1}(y) - \widehat{F}^{-1}(y))dy &= \int_{x=0}^{G_N^{-1}(\alpha)} (\overline{\gamma}(x) - \widehat{\gamma}(x)) g_N(x) dx \\ &= \overline{\Gamma}(G_N^{-1}(\alpha)) - \widehat{\Gamma}(G_N^{-1}(\alpha)) \geq 0,\end{aligned}$$

where the last line comes from the definition of $\overline{\Gamma}$. Theorem 3.8 of (Sriboonchita et al., 2009) therefore implies that $\widehat{F}$ second-order stochastically dominates $\overline{F}$ if and only if the preceding inequality holds for all α. Moreover, it must be that $\overline{\Gamma}(\infty) = \widehat{\Gamma}(\infty)$, since otherwise $\min\{\overline{\Gamma}(x), \widehat{\Gamma}(\infty)\} \circ E^{-1}$ would be a smaller concave function that dominates $\widehat{\Gamma} \circ E^{-1}$. Thus, $\overline{F}$ and $\widehat{F}$ have the same mean, and we conclude that $\widehat{F}$ is a mean-preserving spread of $\overline{F}$. □

Next, we need the following property of the graded gains function:

Lemma 2. *For all $x, y \in \mathbb{R}_+$ with $y \geq x$, $\overline{\gamma}(y) \leq \overline{\gamma}(x) \exp(y - x)$.*

Proof of Lemma 2. Since $\overline{\Gamma} \circ E^{-1}$ is concave, its derivative

$$\frac{\overline{\gamma}(E^{-1}(z)) g_N(E^{-1}(z))}{E'(E^{-1}(z))} = \frac{\overline{\gamma}(E^{-1}(z))}{\exp(E^{-1}(z))}$$

is decreasing. Hence, $\overline{\gamma}(x) \exp(-x)$ is decreasing, which implies the result. □

We can now complete the proof of Proposition 2:

Proof of Proposition 2. From Lemma 1, we know that $\overline{\mathcal{S}}$ is well defined. To complete the proof, we show that $\overline{\Pi}$ is an upper bound on profit.

By the revelation principle, it is sufficient to verify that $\overline{\Pi}$ is an upper bound on revenue for every incentive compatible and interim individually rational direct mechanism (q, t)

(since participation security implies that interim payoffs are non-negative). Fix such a mechanism. Let us write

$$U_i(s_i, s_i') = \int_{\overline{S}_{-i}} (\overline{w}(s_i + \Sigma s_{-i}) q_i(s_i', s_{-i}) - t_i(s_i', s_{-i})) \exp(-\Sigma s_{-i}) ds_{-i},$$

and $U_i(s_i) = U_i(s_i, s_i)$. Incentive compatibility implies that for all i, s_i, and s_i',

$$U_i(s_i) \geq U_i(s_i, s_i') = U_i(s_i') + \int_{\overline{S}_{-i}} (\overline{\gamma}(s_i + \Sigma s_{-i}) - \overline{\gamma}(s_i' + \Sigma s_{-i})) q_i(s_i', s_{-i}) \exp(-\Sigma s_{-i}) ds_{-i},$$

and $U_i(s_i) \geq 0$. Thus, for all $\Delta > 0$,

$$\begin{aligned}
U_i &= \int_{\overline{S}_i} U_i(s_i) \exp(-s_i) ds_i \\
&\geq \int_{\{s \in \overline{S} | s_i \geq \Delta\}} [U_i(s_i - \Delta) + (\overline{\gamma}(\Sigma s) - \overline{\gamma}(\Sigma s - \Delta)) q_i(s_i - \Delta, s_{-i})] \exp(-\Sigma s) ds \\
&= \exp(-\Delta) \int_{\{s \in \overline{S} | s_i \geq \Delta\}} [U_i(s_i - \Delta) + (\overline{\gamma}(\Sigma s) - \overline{\gamma}(\Sigma s - \Delta)) q_i(s_i - \Delta, s_{-i})] \exp(-(\Sigma s - \Delta)) ds \\
&= \exp(-\Delta) \left(U_i + \int_{\overline{S}} (\overline{\gamma}(\Sigma s + \Delta) - \overline{\gamma}(\Sigma s)) q_i(s_i, s_{-i}) \exp(-\Sigma s) ds \right).
\end{aligned}$$

Rearranging, we have:

$$U_i \geq \frac{1}{(\exp(\Delta) - 1)} \int_{\overline{S}} (\overline{\gamma}(\Sigma s + \Delta) - \overline{\gamma}(\Sigma s)) q_i(s_i, s_{-i}) \exp(-\Sigma s) ds.$$

Now, let

$$Q(x) = \frac{1}{g_N(x)} \int_{\{s \in \overline{S} | \Sigma s = x\}} \sum_{i=1}^{N} q_i(s) \exp(-\Sigma s) ds$$

be the expected probability of allocating the good conditional on $\Sigma s = x$. Then

$$\sum_{i=1}^{N} U_i \geq \frac{1}{(\exp(\Delta) - 1)} \int_{x=0}^{\infty} (\overline{\gamma}(x + \Delta) - \overline{\gamma}(x)) Q(x) g_N(x) dx.$$

Since total surplus is

$$\int_{x=0}^{\infty} \overline{\gamma}(x) Q(x) g_N(x) dx,$$

we conclude that an upper bound on profit is

$$\int_{x=0}^{\infty} \left[\overline{\gamma}(x) - \frac{\overline{\gamma}(x + \Delta) - \overline{\gamma}(x)}{\exp(\Delta) - 1} \right] Q(x) g_N(x) dx.$$

By Lemma 2, the term multiplying $Q(x)$ is positive, and since $Q(x) \leq 1$, profit is bounded above by

$$\int_{x=0}^{\infty} \left[\overline{\gamma}(x) - \frac{\overline{\gamma}(x+\Delta) - \overline{\gamma}(x)}{\exp(\Delta) - 1} \right] g_N(x) dx = \int_{x=0}^{\infty} \overline{\gamma}(x) \left[g_N(x) + \frac{g_N(x) - g_N(x-\Delta)}{\exp(\Delta) - 1} \right] dx,$$

where $g_N(x) = 0$ if $x < 0$. The term in brackets converges point-wise for all positive x to $g_N(x) + g'_N(x) = g_{N-1}(x)$ as $\Delta \to 0$. To apply the dominated convergence theorem, all that remains is to present an integrable bounding function, which is done in Lemma 16 in Appendix A. As a result, as $\Delta \to 0$, the profit bound converges to $\overline{\Pi}$. □

This argument is closely related to that of Myerson (1981), but we have used the special structure of $\overline{\mathcal{S}}$ to sidestep the non-trivial technical question of whether the envelope theorem holds for the problem $\max_{s'_i} U_i(s_i, s'_i)$ (cf. Milgrom and Segal, 2002).

4.2.2 Lower bound on profit for $\overline{\mathcal{M}}$

The next result establishes condition 2 in the definition of a strong maxmin solution.

Proposition 3. *$\overline{\mathcal{M}}$ is a well-defined mechanism. For all information structures $\mathcal{S}$ and equilibria β of $(\overline{\mathcal{M}}, \mathcal{S})$, $\Pi(\overline{\mathcal{M}}, \mathcal{S}, \beta) \geq \overline{\Pi}$.*

The first step towards proving Proposition 3 is the following:

Lemma 3. *The aggregate allocation sensitivity $\overline{\mu}$ is decreasing. As a result, $\overline{\lambda}$ is concave.*

Proof of Lemma 3. On a non-graded interval, $\overline{\mu}(x) = (N-1)/x$, which is decreasing, and on a graded interval $[a, b]$, $\overline{\mu}(x) = C(a, b)$. The fact that $\overline{\mu}$ is decreasing across graded intervals then follows from the definition of $C(a, b)$ and the well-known inequality

$$\frac{N-1}{N}\frac{1}{b}(b^N - a^N) \leq b^{N-1} - a^{N-1} \leq \frac{N-1}{N}\frac{1}{a}(b^N - a^N),$$

e.g., Hardy, Littlewood, and Pólya (1934, equation (2.15.2)).

Concavity of $\overline{\lambda}$ then follows from the fact $\overline{\mu}$ is decreasing and equation (18). □

We next verify that $\overline{\mathcal{M}}$ is well-defined.

Lemma 4. *$\overline{\mathcal{M}}$ is a well-defined mechanism, has bounded aggregate transfers $\overline{T}$, and satisfies participation security.*

Proof of Lemma 4. Three properties need to be verified: Feasibility of the allocation rule, existence and boundedness of the transfers, and participation security.

Clearly, $\overline{Q}(x) \geq 0$, since the constants $C(a, b)$ and $D(a, b)$ are positive. We now argue that $\overline{Q}(x) \leq 1$. This is clearly true at the end points of a graded interval. Moreover, on a graded interval $[a, b]$,

$$\overline{Q}'(x) = \frac{C(a, b)}{N} - (N-1)\frac{D(a, b)}{x^N},$$

which is increasing. Thus, $\overline{Q}$ is convex on $[a,b]$, and $\overline{Q}(x) \leq \max\{\overline{Q}(a), \overline{Q}(b)\} = 1$.[29]

To show that $\overline{T}$ is well defined and finite, we first show that $\overline{\lambda}$ defined in (18) is bounded: The last integral in (18) is bounded above by

$$\int_{v=\underline{v}}^{\overline{v}} \overline{\mu}\left(G_N^{-1}(H(v))\right) dv = \int_{y=0}^{\infty} \overline{\mu}(y)\widehat{w}(dy). \tag{20}$$

From part one of the left-tail assumption, there exists $\epsilon > 0$ such that if $x \leq \epsilon$, $(\widehat{w}(x) - \underline{v})/x^{\varphi} \leq 1$ for some $\varphi > 1$. If the value function is not graded at x, $\overline{\mu}(x) = (N-1)/x$, and if x is in a graded interval $[a,b]$, then

$$\overline{\mu}(x) = C(a,b) = \frac{b^N - ba^{N-1}}{b^N - a^N}\frac{N}{b} \leq \frac{N}{b} \leq \frac{N}{x}. \tag{21}$$

Thus, if $x \leq \epsilon$, we can plug in the bound and integrate by parts to obtain

$$\begin{aligned}
\int_{y=0}^{\infty} \overline{\mu}(y)\widehat{w}(dy) \leq \int_{y=0}^{\infty} \frac{N}{y}\widehat{w}(dy) &= \int_{y=0}^{\epsilon} \frac{N}{y^2}(\widehat{w}(y) - \underline{v})dy + \int_{y=\epsilon}^{\infty} \frac{N}{y^2}(\widehat{w}(y) - \underline{v})dy \\
&\leq N\int_{y=0}^{\epsilon} y^{\varphi-2}dy + \int_{y=\epsilon}^{\infty} \frac{N}{y^2}(\overline{v} - \underline{v})dy \\
&= N\frac{1}{\varphi - 1}\epsilon^{\varphi-1} + N\frac{\overline{v} - \underline{v}}{\epsilon}.
\end{aligned}$$

Hence, the last integral in the definition of $\overline{\lambda}(v)$ is bounded. The middle integral is simply the expectation of the last integral across lower bounds $v \sim H$, so we conclude that $\overline{\lambda}(v)$ is bounded.

Since $\overline{\lambda}$ is bounded and $\overline{\mu}(x) \leq N/x$, $\overline{\Xi}(x)$ goes to infinity as $x \to 0$ at a rate no faster than $1/x$, so $\overline{\Xi}(x)g_N(x)$ is integrable on $[0,\infty)$. This shows that $\overline{T}$ in (12) is well defined. Moreover, equation (12) clearly shows that $\overline{T}$ is continuous at $x > 0$, so to show that $\overline{T}$ is bounded, it suffices to show that $\lim_{x\to\infty} \overline{T}(x) < \infty$ and $\lim_{x\to 0} \overline{T}(x) < \infty$.

By Lemma 11 in Appendix A, we have $0/0$ for $\overline{T}(x)$ as $x \to \infty$. By L'Hôpital's rule, we have

$$\lim_{x\to\infty} \overline{T}(x) = \lim_{x\to\infty} \frac{\overline{\Xi}(x)g_N(x)}{g_{N-1}(x) - g_N(x)} = \lim_{x\to\infty} \frac{\overline{\Xi}(x)}{\frac{N-1}{x} - 1} = \overline{\lambda}(\overline{v}) + c < \infty,$$

since $\lim_{x\to\infty} \overline{\mu}(x) = 0$ by (21), and $\lim_{x\to\infty} \overline{Q}(x) = 1$ by Lemma 13 in Appendix A.

For $x \to 0$, we apply L'Hôpital's rule again:

$$\lim_{x\to 0} \overline{T}(x) = \lim_{x\to 0} \frac{\overline{\Xi}(x)g_N(x)}{g_{N-1}(x) - g_N(x)} = \lim_{x\to 0} \frac{\overline{\Xi}(x)\frac{x}{N-1}}{1 - \frac{x}{N-1}} = \underline{v}\overline{Q}(0) < \infty,$$

where we use the fact that $\overline{\lambda}$ and $\overline{Q}$ are bounded, and Lemma 15 in Appendix A shows that there is either a graded interval at 0 in which case $\lim_{x\to 0} \widehat{w}(x)\overline{\mu}(x)\frac{x}{N-1} = 0 = \underline{v}\overline{Q}(0)$, or there is a non-graded interval at 0 in which case $\lim_{x\to 0} \widehat{w}(x)\overline{\mu}(x)\frac{x}{N-1} = \underline{v} = \underline{v}\overline{Q}(0)$.

[29] We thank a referee for suggesting this argument.

Thus, $\overline{T}$ is bounded. The above argument also shows that $\overline{T}$ is continuous at $x = 0$ as well.

Finally, participation security follows from the definition of $\overline{T}$, which implies that $\bar{t}_i(0, m_{-i}) = \underline{v}\overline{q}_i(0, m_{-i})$. □

We now develop the lower bound on profit in $\overline{\mathcal{M}}$.

Lemma 5. *At all $m \neq 0$, $\overline{q}_i(m)$ is right-differentiable in m_i and $\nabla \cdot \overline{q}(m) = \overline{\mu}(\Sigma m)$.*

Proof of Lemma 5. Suppose $m \neq 0$. For the right-differentiability of $\overline{q}_i$ it suffices to show that $\overline{Q}(x)$ is right differentiable at every $x > 0$. There are three cases.

Case 1: There exists an $\epsilon > 0$ such that $[x, x+\epsilon]$ is a subset of a graded interval. From the formula of $\overline{Q}$ in (16) it is immediate that $\overline{Q}$ is right differentiable at x, and in fact

$$\overline{Q}'(x) = \overline{\mu}(x) - \frac{N-1}{x}\overline{Q}(x). \tag{22}$$

Case 2: There exists an $\epsilon > 0$ such that $\overline{\Gamma}(y) = \widehat{\Gamma}(y)$ for all $y \in [x, x+\epsilon]$. Then $\overline{Q}(y) = 1$ and $\overline{\mu}(y) = (N-1)/y$ for all $y \in [x, x+\epsilon]$, so $\overline{Q}$ is again right differentiable at x, and equation (22) again holds.

Case 3: For every $\epsilon > 0$, there exist $x', x'' \in (x, x+\epsilon]$ such that x' is not graded and x'' is graded: $\overline{\Gamma}(x') = \widehat{\Gamma}(x')$ and $\overline{\Gamma}(x'') > \widehat{\Gamma}(x'')$. This implies that $\overline{Q}(x) = 1$, for otherwise x is in the interior of a graded interval and we are in Case 1. Let $\{x_n\}$ be a sequence converging to x from the right. We show that for every $\epsilon > 0$, there exists a $\overline{n}$ such that for all $n \geq \overline{n}$, we have

$$\left| \frac{\overline{Q}(x_n) - \overline{Q}(x)}{x_n - x} \right| \leq \epsilon, \tag{23}$$

i.e., the $\overline{Q}$ is right differentiable at x, and $\overline{Q}'(x) = 0$. This again implies (22), since $\overline{\mu}(x) = (N-1)/x$ by (17).

Given $\epsilon > 0$, choose $\epsilon' > 0$ so that

$$\frac{N-1}{x} - \frac{N-1}{x+\epsilon'}\left(1 - \frac{2(N-1)}{x}\epsilon'\right) \leq \epsilon, \qquad \frac{N-1}{x+\epsilon'} - \frac{N-1}{x} \geq -\epsilon.$$

By assumption, there exists an $x' \in (x, x+\epsilon']$ at which the gains function is not graded. Choose $\overline{n}$ so that $x_n \in [x, x']$ for all $n \geq \overline{n}$.

Consider any $n \geq \overline{n}$. Suppose x_n is graded and $\overline{Q}(x_n) < 1$ (for otherwise (23) is trivial). Let $[a, b]$ be the graded interval containing x_n. We have $x < a < x_n < b \leq x' \leq x + \epsilon'$: we have $b \leq x'$ because x' is not graded and $x < a$ because there are other non-graded points in $(x, x_n]$. Since $\overline{Q}(a) = \overline{Q}(x) = 1$, we have:

$$\frac{\overline{Q}(x_n) - \overline{Q}(x)}{x_n - x} = \frac{x_n - a}{x_n - x}\frac{\overline{Q}(x_n) - \overline{Q}(a)}{x_n - a} = \frac{x_n - a}{x_n - x}\overline{Q}'(y) = \frac{x_n - a}{x_n - x}\left(\overline{\mu}(y) - \frac{N-1}{y}\overline{Q}(y)\right)$$

for some $y \in (a, x_n)$ by the mean value theorem and equation (22) applied to the graded interval $[a, b]$. Equation (22) also implies that $\overline{Q}'(z) \leq \frac{2(N-1)}{x}$ for any $z \in [a, b]$, so $|\overline{Q}(y) -$

$1| \leq \frac{2(N-1)}{x}\epsilon'$; and since both $\overline{\mu}(y)$ and $(N-1)/y$ are in the interval $((N-1)/b, (N-1)/a)$, we have

$$-\epsilon \leq \frac{N-1}{x+\epsilon'} - \frac{N-1}{x} \leq \frac{\overline{Q}(x_n) - \overline{Q}(x)}{x_n - x} \leq \frac{N-1}{x} - \frac{N-1}{x+\epsilon'}\left(1 - \frac{2(N-1)}{x}\epsilon'\right) \leq \epsilon,$$

which proves (23).

Finally, it is easy to check that $\nabla \cdot \overline{q}(m) = \overline{\mu}(\Sigma m)$ by applying the product rule and equation (22) to the definition of $\overline{q}$. □

Lemma 6. *The ex ante expectation of $\overline{\lambda}(v)$ is $\overline{\Pi}$:*

$$\int_{v=\underline{v}}^{\overline{v}} \overline{\lambda}(v)H(dv) = \int_{x=0}^{\infty} \overline{\lambda}(\widehat{w}(x))g_N(x)dx = \overline{\Pi}.$$

Proof of Lemma 6. The first equality follows from the change of variables $v = \widehat{w}(x) = H^{-1}(G_N(x))$. For the second equality, it is sufficient to show that the middle integral in (18) is equal to the ex ante expectation of the last integral, which using (20) and Tonelli's theorem is:

$$\int_{x=0}^{\infty}\int_{y=0}^{x} g_N(y)dy\overline{\mu}(x)\widehat{w}(dx) = \int_{x=0}^{\infty} \overline{\mu}(x)G_N(x)\widehat{w}(dx).$$

□

Lemma 7. *For every $m \neq 0$, $\overline{t}_i(m)$ is right differentiable with respect to m_i, and*

$$\nabla \cdot \overline{t}(m) - \Sigma\overline{t}(m) = \overline{\Xi}(\Sigma m).$$

Proof of Lemma 7. We first prove that $\overline{t}_i(m)$ is right differentiable with respect to m_i at $m \neq 0$. By Lemma 5, it suffices to show that $\overline{T}$ is right differentiable at every $x > 0$, and for that it suffices to show that $\overline{\Xi}$ is right continuous at every $x > 0$.

To that end, first note that $\overline{\mu}$ is right-continuous at every $x > 0$. This is clear from the definition of $\overline{\mu}$ in equation (17) when x is the left-end point or in the interior of a graded interval, or when x is the left-end point or in the interior of a non-graded interval. The only other case is when x is a limit point both of graded points and of non-graded points. In this case, $\overline{\mu}(x)$ is defined to be $(N-1)/x$ by (17), and for any sequence $\{x_n\}$ converging to x from the right, $\overline{\mu}(x_n)$ converges to $\overline{\mu}(x) = (N-1)/x$: The reason is that for any $\epsilon > 0$, there exists a non-graded point $x' \in (x, x+\epsilon)$. Hence, when n is sufficiently large, $x_n \in [x, x']$ so $\overline{\mu}(x_n) \in [(N-1)/(x+\epsilon), (N-1)/x]$ as $\overline{\mu}$ is decreasing from Lemma 3.

Next, we prove right continuity of $\overline{\Xi}$ at $x > 0$. Since $\overline{Q}(x)$ is right continuous at all $x > 0$ by the proof of Lemma 5, the obstacle to right-continuity of $\overline{\Xi}(x)$ can only come from $\overline{\mu}(x)\widehat{w}(x)$ or $\overline{\lambda}(\widehat{w}(x))$. If $\widehat{w}$ is discontinuous at $x > 0$, then $\widehat{\Gamma}$ has a convex kink at x, so x must be in the interior of a graded interval $[a, b]$. Since $\overline{\mu}$ is continuous (in fact, constant) in (a, b), $\overline{\Xi}$ is continuous at x as the discontinuity in $\overline{\mu}(y)\widehat{w}(y)$ at $y = x$ is exactly canceled by the discontinuity in $\overline{\lambda}(\widehat{w}(y))$ at $y = x$ (recall that $\overline{\lambda}(\widehat{w}(y)) = C + \int_{z=0}^{y} \overline{\mu}(z)d\widehat{w}(z)$ for a constant C). On the other hand, if $\widehat{w}$ is continuous at $x > 0$, then $\overline{\Xi}$ is right-continuous

at x because $\overline{\mu}$ is right-continuous there by the previous paragraph. We conclude that $\overline{\Xi}$ is right-continuous, and hence $\overline{T}$ is right-differentiable, at $x > 0$.

Finally, interpreting d/dx as the right-derivative, we have $\frac{d}{dx}\left(g_N(x)\overline{T}(x)\right) = g_N(x)\overline{\Xi}(x)$, so

$$\left(\frac{N-1}{x} - 1\right)\overline{T}(x) + \overline{T}'(x) = \overline{\Xi}(x), \tag{24}$$

which proves the lemma since by the product rule $\nabla \cdot \bar{t}(m) = \frac{N-1}{\Sigma m}\overline{T}(\Sigma m) + \overline{T}'(\Sigma m)$. □

Lemma 8. *For any information structure $\mathcal{S}$ and equilibrium β of $(\overline{\mathcal{M}}, \mathcal{S})$,*

$$\int_S \int_{\overline{M}} \left[w(s)\nabla \cdot \bar{q}(m) - \nabla \cdot \bar{t}(m)\right] \beta(dm|s)\pi(ds) \leq 0. \tag{25}$$

This result says that in any equilibrium, local upward deviations must not be attractive. If a bidder were to marginally increase all of the messages they send in equilibrium, the change in payoff would be

$$\int_S \int_{\overline{M}} \left(w(s)\frac{\partial}{\partial m_i}\bar{q}_i(m) - \frac{\partial}{\partial m_i}\bar{t}_i(m)\right) \beta(dm|s)\pi(ds) \leq 0.$$

Summing across i gives (25). A technical complication is that the allocation sensitivity may blow up as the aggregate bid goes to zero, so that deriving this constraint as a limit of non-local deviations is not trivial. We resolve this technical complication by appealing to part 2 of the left-tail assumption; a detailed proof can be found in Appendix A.

We can now complete the proof of Proposition 3.

Proof of Proposition 3. We have already argued in Lemma 4 that $\overline{\mathcal{M}}$ is well-defined. To complete the proof, it suffices to show that profit in any equilibrium in any information structure is at least $\overline{\Pi}$. This is established in two steps.

Step 1: For any v and x,

$$\begin{aligned}
\overline{\lambda}(v) &= \overline{\lambda}(\widehat{w}(x)) - \int_{\nu=v}^{\widehat{w}(x)} \overline{\mu}(G_N^{-1}(H(\nu)))d\nu \\
&\leq \overline{\lambda}(\widehat{w}(x)) - (\widehat{w}(x) - v)\overline{\mu}(x) \\
&= v\,\overline{\mu}(x) - \overline{\Xi}(x) - c\,\overline{Q}(x),
\end{aligned}$$

where the second line follows from the fact that $\overline{\mu}$ is decreasing (Lemma 3), and the third line follows from the definition of $\overline{\Xi}$.

Step 2: Fix an information structure $\mathcal{S}$. Profit in an equilibrium β of $(\overline{\mathcal{M}}, \mathcal{S})$ is

$$\int_S \int_{\overline{M}} \left[\overline{T}(\Sigma m) - c\,\overline{Q}(\Sigma m)\right] \beta(dm|s)\pi(ds).$$

By Lemma 8, this is at least

$$\int_S \int_{\overline{M}} \left[w(s)\,\nabla \cdot \bar{q}(m) - \left(\nabla \cdot \bar{t}(m) - \Sigma\bar{t}(m)\right) - c\,\overline{Q}(\Sigma m)\right] \beta(dm|s)\pi(ds)$$

$$
\begin{aligned}
&= \int_S \int_{\overline{M}} \left[w(s)\, \overline{\mu}(\Sigma m) - \overline{\Xi}(\Sigma m) - c\, \overline{Q}(\Sigma m) \right] \beta(dm|s)\pi(ds) \\
&\geq \int_S \overline{\lambda}(w(s))\pi(ds) \\
&\geq \int_V \overline{\lambda}(v) H(dv).
\end{aligned}
$$

The second line follows from Lemmas 5 and 7. The third line follows from Step 1. The last inequality uses concavity of $\overline{\lambda}$ (Lemma 3), the fact that H is a mean-preserving spread of the distribution of $w(s)$, and Jensen's inequality. The final integral is equal to $\overline{\Pi}$ by Lemma 6. □

4.2.3 Truth-telling equilibrium

We now come to the last condition for $(\overline{\mathcal{M}}, \overline{\mathcal{S}}, \overline{\beta})$ to be a strong maxmin solution.

Proposition 4. *The truthful strategies $\overline{\beta}$ are an equilibrium of the game $(\overline{\mathcal{M}}, \overline{\mathcal{S}})$.*

Proof of Proposition 4. Let

$$
U_i(m_i, m_i') = \int_{\overline{M}_{-i}} \left[\overline{w}(m_i + \Sigma m_{-i}) \overline{q}_i(m_i', m_{-i}) - \overline{t}_i(m_i', m_{-i}) \right] \exp(-\Sigma m_{-i})\, dm_{-i}
$$

denote the interim expected utility from reporting m_i' when the true signal is m_i and others report truthfully. We will show that the difference $U_i(m_i, m_i) - U_i(m_i, m_i')$ is non-negative for all i, m_i, and m_i'.

We first derive a convenient expression for the interim expected transfer:[30]

$$
\int_{\overline{M}_{-i}} \overline{t}_i(m_i', m_{-i}) \exp(-\Sigma m_{-i}) dm_{-i}
$$

[30]The final expression for the interim transfer substantiates the claim in Section 3 that boundedness of the transfers is a sufficient condition for incentive compatibility of $\overline{q}$ on $\overline{\mathcal{S}}$. We can rewrite the transfer as

$$
t_i(m) = \exp(m_i) \left(t_i(0, m_{-i}) + \int_{x=0}^{m_i} \xi_i(x, m_{-i}) \exp(-x) dx \right),
$$

where $\xi_i(m) = \partial t_i(m)/dm_i - t_i(m)$ is bidder i's individual excess growth, and $\Sigma\xi = \overline{\Xi}$ (cf. (8)). Since $\overline{q}_i(0, m_{-i}) = 0$ for $m_{-i} \neq 0$, we restrict attention to $t_i(0, m_{-i}) = 0$. Boundedness of i's transfer then implies that

$$
\int_{x=0}^{\infty} \xi_i(x, m_{-i}) \exp(-x) dx = 0 \tag{26}
$$

for $m_{-i} \neq 0$, in which case we can rewrite

$$
t_i(m) = -\int_{x=0}^{\infty} \xi_i(m_i + x, m_{-i}) \exp(-x) dx.
$$

But if we take the expectation of $\xi_i(m_i + x, m_{-i})$ over m_{-i}, equation (26), combined with $\Sigma\xi = \overline{\Xi}$, implies that the interim transfer is exactly as given. Moreover, when $N = 2$, this identity holds only if (26) holds as well. For more than two bidders, incentive compatibility of $\overline{q}$ on $\overline{\mathcal{S}}$ is equivalent to, for all i and m_i, $\int_{\overline{M}_{-i}} (\xi_i(m_i, m_{-i}) - \overline{\Xi}(m_i, m_{-i})) \exp(-\Sigma m_{-i}) dm_{-i} = 0$.

$$
\begin{aligned}
&= \int_{x=0}^{\infty} \frac{m_i'}{m_i'+x} \overline{T}(m_i'+x) g_{N-1}(x) dx \\
&= -\int_{x=0}^{\infty} \left(\frac{N-1}{m_i'+x} g_N(x) - g_{N-1}(x) \right) \overline{T}(m_i'+x) dx \\
&= -\int_{x=0}^{\infty} \left(\left(\frac{N-1}{m_i'+x} - 1 \right) \overline{T}(m_i'+x) + \overline{T}'(m_i'+x) \right) g_N(x) dx \\
&= -\int_{x=0}^{\infty} \overline{\Xi}(m_i'+x) g_N(x) dx,
\end{aligned}
$$

where the first equality is from the definition of $\bar{t}$ and the fact that $m_{-i} = 0$ occurs with zero probability; the second rearranges the formula for $(m_i' g_{N-1}(x))/(m_i'+x)$; the third is obtained by integrating $\int \overline{T}(m_i'+x) g_N'(x) dx$ by parts and the fact that $\overline{T}$ is bounded (Lemma 4); and the fourth substitutes using equation (24).

We next compute the interim expected payoff from the allocation:

$$
\begin{aligned}
&\int_{\overline{M}_{-i}} \overline{w}(m_i + \Sigma m_{-i}) \overline{q}_i(m_i', m_{-i}) \exp(-\Sigma m_{-i}) dm_{-i} \\
&= \int_{x=0}^{\infty} \overline{w}(m_i+x) \frac{m_i'}{m_i'+x} \overline{Q}(m_i'+x) g_{N-1}(x) dx \\
&= \int_{x=0}^{\infty} \overline{w}(m_i+x) \left[\overline{Q}(m_i'+x) - \frac{x}{m_i'+x} \overline{Q}(m_i'+x) \right] g_{N-1}(x) dx \\
&= \int_{x=0}^{\infty} \overline{w}(m_i+x) \left[\overline{Q}(m_i'+x) - \frac{x}{N-1} \left(\overline{\mu}(m_i'+x) - \overline{Q}'(m_i'+x) \right) \right] g_{N-1}(x) dx \\
&= \int_{x=0}^{\infty} \overline{w}(m_i+x) \left[\overline{Q}(m_i'+x) g_{N-1}(x) - \left(\overline{\mu}(m_i'+x) - \overline{Q}'(m_i'+x) \right) g_N(x) \right] dx \\
&= \int_{x=0}^{\infty} \left[\overline{w}(m_i+x) (\overline{Q}(m_i'+x) g_{N-1}(x) - \overline{\mu}(m_i'+x) g_N(x)) dx - \overline{Q}(m_i'+x) d(\overline{w}(m_i+x) g_N(x)) \right] \\
&= \int_{x=0}^{\infty} \left[\overline{Q}(m_i'+x) g_N(x) (\overline{w}(m_i+x) dx - \overline{w}(m_i+dx)) - \overline{\mu}(m_i'+x) \overline{w}(m_i+x) g_N(x) dx \right].
\end{aligned}
$$

The first equality is the definition of $\overline{q}$ and the fact that $m_{-i} = 0$ has zero probability; the second rearranges the term $m_i'/(m_i'+x)$; the third substitutes in using (22); the fourth rearranges terms; the fifth integrates $\int \overline{Q}'(m_i'+x) \overline{w}(m_i+x) g_N(x) dx$ by parts; and the last equality applies the product rule (and the fact that $g_N' = g_{N-1} - g_N$) and rearranges terms.

We now use these expressions to compute the interim expected loss from deviating:

$$
\begin{aligned}
&U_i(m_i, m_i) - U_i(m_i, m_i') \\
&= \int_{x=0}^{\infty} (\overline{Q}(m_i+x) - \overline{Q}(m_i'+x)) g_N(x) \left[\overline{w}(m_i+x) dx - \overline{w}(m_i+dx) \right] \\
&\quad + \int_{x=0}^{\infty} \left[(\overline{\mu}(m_i'+x) - \overline{\mu}(m_i+x)) \overline{w}(m_i+x) + \overline{\Xi}(m_i+x) - \overline{\Xi}(m_i'+x) \right] g_N(x) dx.
\end{aligned}
$$

Observe that

$$
\overline{\Xi}(m_i+x) - \overline{\Xi}(m_i'+x)
$$

$$
\begin{aligned}
&= \overline{\mu}(m_i+x)\widehat{w}(m_i+x) - \overline{\mu}(m_i'+x)\widehat{w}(m_i'+x) - \overline{\lambda}(\widehat{w}(m_i+x)) + \overline{\lambda}(\widehat{w}(m_i'+x)) \\
&\qquad - c(\overline{Q}(m_i+x) - \overline{Q}(m_i'+x)) \\
&= \overline{\mu}(m_i+x)\widehat{w}(m_i+x) - \overline{\mu}(m_i'+x)\widehat{w}(m_i'+x) - \int_{y=m_i'+x}^{m_i+x} \overline{\mu}(y)\widehat{w}(dy) \\
&\qquad - c(\overline{Q}(m_i+x) - \overline{Q}(m_i'+x)) \\
&= \int_{y=m_i'+x}^{m_i+x} \widehat{w}(y)\overline{\mu}(dy) - c(\overline{Q}(m_i+x) - \overline{Q}(m_i'+x)) \\
&= \int_{y=m_i'+x}^{m_i+x} \overline{w}(y)\overline{\mu}(dy) - c(\overline{Q}(m_i+x) - \overline{Q}(m_i'+x)).
\end{aligned}
$$

The first equality uses the definition of $\overline{\Xi}$, equation (19); the second uses the definition of $\overline{\lambda}$, equation (18); the third is integration by parts; and the last equality uses the fact that $\overline{\mu}(y)$ is constant on the interior of graded intervals, so that the measure $\overline{\mu}(dy)$ assigns zero mass to the points where $\widehat{w} \neq \overline{w}$.

Substituting this last expression into the loss from deviating, we obtain

$$
\begin{aligned}
&U_i(m_i, m_i) - U_i(m_i, m_i') \\
&= \int_{x=0}^{\infty} (\overline{Q}(m_i+x) - \overline{Q}(m_i'+x)) g_N(x) \left[(\overline{w}(m_i+x) - c)dx - \overline{w}(m_i+dx)\right] \\
&\qquad + \int_{x=0}^{\infty} \left[(\overline{\mu}(m_i'+x) - \overline{\mu}(m_i+x))\overline{w}(m_i+x) + \int_{y=m_i'+x}^{m_i+x} \overline{w}(y)\overline{\mu}(dy) \right] g_N(x)dx.
\end{aligned}
$$

Lemma 2 implies that the measure

$$
(\overline{w}(m_i+x) - c)dx - \overline{w}(m_i+dx) = \overline{\gamma}(m_i+x)dx - \overline{\gamma}(m_i+dx)
$$

is non-negative and its support consists of the non-graded intervals, on which $\overline{Q}(m_i+x) = 1 \geq \overline{Q}(m_i'+x)$, so that the first integral is non-negative. The second integral is also non-negative, because $\overline{w}$ is increasing, so

$$
\int_{y=m_i'+x}^{m_i+x} \overline{w}(y)\overline{\mu}(dy) \geq \overline{w}(m_i+x)(\overline{\mu}(m_i+x) - \overline{\mu}(m_i'+x)).
$$

We conclude that $U_i(m_i, m_i) \geq U_i(m_i, m_i')$, as desired. □

Theorem 1 follows from Propositions 2, 3, and 4.

4.3 The must-sell case

We now discuss the variant of our model where the good must be sold. All of our tools still apply and almost immediately give us the solution.

A *must-sell mechanism* is a mechanism for which $\Sigma q(m) = 1$ for all m. A *must-sell strong maxmin solution* is a triple $(\mathcal{M}, \mathcal{S}, \beta)$ satisfying conditions 1–3 in Section 2.5, but

where $\mathcal{M}$ is a must-sell mechanism and condition 1 only has to hold for $\mathcal{M}'$ that are must-sell mechanisms.

Let $\widehat{\mathcal{S}}$ be the information structure where signals are i.i.d. standard exponential and the value function is $\widehat{w}$. Also, let $\widehat{\mathcal{M}}$ be the proportional auction with allocation

$$\widehat{q}_i(m) = \begin{cases} \frac{1}{N} & \text{if } \Sigma m = 0; \\ \frac{m_i}{\Sigma m} & \text{if } \Sigma m > 0, \end{cases}$$

and transfers

$$\widehat{t}_i(m) = \begin{cases} \frac{1}{N}\widehat{T}(\Sigma m) & \text{if } \Sigma m = 0; \\ \frac{m_i}{\Sigma m}\widehat{T}(\Sigma m) & \text{if } \Sigma m > 0. \end{cases}$$

We define $\widehat{\lambda}$, $\widehat{\Xi}$ and $\widehat{T}$ according to analogous formulae as those for $\overline{\lambda}$, $\overline{\Xi}$, and $\overline{T}$, using $\widehat{w}$, $\widehat{\mu}(x) = (N-1)/x$, and $\widehat{Q}(x) = 1$ in place of $\overline{w}$, $\overline{\mu}$, and $\overline{Q}$. Let

$$\widehat{\Pi} = \int_{x=0}^{\infty} \widehat{\gamma}(x) g_{N-1}(x)\, dx. \tag{27}$$

Finally, let $\widehat{\beta} = \overline{\beta}$.

Theorem 2 (Must-sell solution)**.** *$(\widehat{\mathcal{M}}, \widehat{\mathcal{S}}, \widehat{\beta})$ is a must-sell strong minmax solution with a profit guarantee of $\widehat{\Pi}$ defined by* (27)*.*

Proof of Theorem 2. The proof of Proposition 2 goes through with $\widehat{\gamma}$ in place of $\overline{\gamma}$, except that we do not need to invoke Lemma 2 (which does not hold for the fully-revealing gains function) to conclude that the profit upper bound is maximized by setting $Q(x) = 1$. Instead, the conclusion follows directly from the must-sell assumption, so that (27) is an upper bound on profit in $\widehat{\mathcal{S}}$.

The proof of Proposition 3 remains valid with $\widehat{\gamma}$ in place of $\overline{\gamma}$. (In the proof of Lemma 7, $\widehat{\mu}$ is continuous everywhere, in particular where $\widehat{w}$ is discontinuous).

Proposition 4 also goes through, with the only modification being that the last term in the deviation payoff involving $\overline{Q}$ disappears, so that again we do not need to invoke Lemma 2. Thus, the mechanism $\widehat{\mathcal{M}}$ guarantees the Seller at least $\widehat{\Pi}$ in any equilibrium, and $\widehat{\beta}$ is an equilibrium of the game $(\widehat{\mathcal{M}}, \widehat{\mathcal{S}})$. □

It is sometimes possible to use the simple formulas for $\widehat{\mu}$ and $\widehat{Q}$ to simplify the must-sell aggregate transfer. For example, Appendix D.1 shows that when v is standard uniform,

$$\widehat{T}(x) = G_{N-1}(x) + \frac{\binom{2N-3}{N-1}}{2^{2N-3}} \frac{G_N(x) - G_{2N-2}(2x)}{g_N(x)}.$$

When $N = 2$, this further simplifies to $\widehat{T}(x) = (1 - (1 - e^{-x})/x)/2$. We shall describe the many-bidder limit of this formula in Section 6.

4.4 Single-crossing distributions

We now discuss a class of distributions for which the maxmin aggregate allocation is relatively simple. The distribution H is *single crossing* if there is a cutoff $\overline{x}$ such that $\widehat{\Gamma} \circ E^{-1}$ is convex on $[0, \overline{x}]$ and concave on $[\overline{x}, \infty)$. When the gains function is differentiable, this is equivalent to saying that $\widehat{\gamma}(x) - \widehat{\gamma}'(x)$ is single crossing from below at $x = E^{-1}(\overline{x})$. This is in a sense a counterpart to the regular case of Myerson (1981), under which the Seller only has an incentive to ration the good when signals are a below a cutoff.

If H is single crossing, then there is a single graded interval, denoted $[0, x^*]$, on which $\overline{Q}(x) = xC(0, x^*)/N = x/x^*$ (since $D(0, x^*) = 0$). The maxmin allocation is therefore $\overline{q}_i(m_i, m_{-i}) = m_i / \max\{x^*, \Sigma m\}$. We can interpret m_i as bidder i's demand for the good in units where the aggregate supply is x^*. The allocation rule simply says that the bidders get their demands if the aggregate demand is feasible, and the good is rationed proportionally otherwise.

The uniform distribution is single crossing for all N. To see this, observe that the fully-revealing gains function is $\widehat{\gamma}(x) = G_N(x) - c$, so $\widehat{\gamma}(x) - \widehat{\gamma}'(x) = G_N(x) - c - g_N(x)$. This is $-c$ when $x = 0$, and its derivative is

$$2g_N(x) - g_{N-1}(x) = \left(\frac{2x}{N-1} - 1\right) g_{N-1}(x),$$

so that $\widehat{\gamma}(x) - \widehat{\gamma}'(x)$ is decreasing for $x < (N-1)/2$ and increasing otherwise, which implies that it crosses zero once, from below. Thus, $\overline{\gamma} = \overline{\gamma}(0)\exp(x)$ on $[0, x^*]$, and it is fully revealing above x^*. For these segments to meet continuously, it must be that $\overline{\gamma}(0) = \exp(-x^*)(G_N(x^*) - c)$. This implies that x^* is the unique positive solution to[31]

$$\int_{x=0}^{x^*} (G_N(x) - c) g_N(x) dx = \exp(-x^*)(G_N(x^*) - c) \int_{x=0}^{x^*} \exp(x) g_N(x) dx. \tag{28}$$

Maxmin profit is

$$\overline{\Pi} = \int_{x=0}^{x^*} \overline{\gamma}(0)\exp(x) g_{N-1}(x) dx + \int_{x=x^*}^{\infty} (G_N(x) - c) g_{N-1}(x) dx,$$

while maxmin profit among must-sell mechanisms is only

$$\widehat{\Pi} = \int_{x=0}^{\infty} (G_N(x) - c) g_{N-1}(x) dx.$$

This example is continued in Section 6. We also give an explicit formula for $\overline{T}$ in Appendix D.2 when $c = 0$.

[31] Both sides of (28) are zero at $x^* = 0$, and the difference between the left and right has a derivative with respect to x^* that has the same sign as $G_N(x^*) - c - g_N(x^*)$, which as we have argued is single crossing from below.

5 Uniqueness and finite approximability

We have constructed a particular strong maxmin solution. We know for a fact that there are others (and we will comment further on this shortly). However, as we now show, any sufficiently well-behaved solution must have the same profit guarantee of $\overline{\Pi}$. As part of developing this result, we will show that the mechanism $\overline{\mathcal{M}}$ and information structure $\overline{\mathcal{S}}$ can both be approximated with finite objects that guarantee profit arbitrarily close to $\overline{\Pi}$. A further implication of this result is that $\overline{\mathcal{M}}$ and $\overline{\mathcal{S}}$ are limits of ϵ-equilibria of standard zero-sum games, where the Seller and Nature choose finite mechanisms and finite information structures, respectively. These results provide additional foundations for the strong maxmin solution that we construct.

Let us say that a mechanism is *finite* if M_i is finite for all i. Finite message sets can be identified with subsets of $\mathbb{N}$, so that the set of finite mechanisms exists and is denoted by $\mathbf{M}^F$. Similarly, an information structure is *finite* if S_i is finite for all i, and $\mathbf{S}^F$ is the set of finite information structures.

A strong maxmin solution $(\mathcal{M}, \mathcal{S}, \beta)$ with profit guarantee Π is *finitely approximable* if for any $\epsilon > 0$,

(i) there exists $\mathcal{M}^F \in \mathbf{M}^F$ such that for any information structure $\mathcal{S}'$ and equilibrium β' of $(\mathcal{M}^F, \mathcal{S}')$, $\Pi(\mathcal{M}^F, \mathcal{S}', \beta') \geq \Pi - \epsilon$;

(ii) there exists $\mathcal{S}^F \in \mathbf{S}^F$ such that for any mechanism $\mathcal{M}'$ and equilibrium β' of $(\mathcal{M}', \mathcal{S}^F)$, $\Pi(\mathcal{M}', \mathcal{S}^F, \beta') \leq \Pi + \epsilon$.

In other words, a solution is finitely approximable if there are finite mechanisms and finite information structures that guarantee profit close to Π for the Seller and Nature, respectively.

Theorem 3 (Finite approximability)**.** *The solution $(\overline{\mathcal{M}}, \overline{\mathcal{S}}, \overline{\beta})$ is finitely approximable.*

The theorem follows from two propositions, whose proofs are in Appendix B. Given a non-negative real number $\underline{m}$ and a positive integer K, let $\overline{\mathcal{M}}(\underline{m}, K)$ be the mechanism where each bidder's message space is $\{\underline{m} + l/K | l = 0, \ldots, K^2\}$, the allocations are the restriction of $\overline{q}$ to this message space, and transfers are given by $t_i(m) = \overline{t}_i(m) - L_p\underline{m}$, where L_p is a Lipschitz constant for the premium aggregate transfer $\overline{T} - \underline{v}\overline{Q}$.[32]

The purpose of the discount $L_p\underline{m}$ is to satisfy participation security, since

$$\begin{aligned} t_i(\underline{m}, m_{-i}) &= \frac{\underline{m}}{\underline{m} + \Sigma m_{-i}} \overline{T}(\underline{m} + \Sigma m_{-i}) - L_p\underline{m} \\ &\leq \frac{\underline{m}}{\underline{m} + \Sigma m_{-i}} \left(\underline{v}\overline{Q}(\underline{m} + \Sigma m_{-i}) + L_p(\underline{m} + \Sigma m_{-i})\right) - L_p\underline{m} \\ &\leq \frac{\underline{m}}{\underline{m} + \Sigma m_{-i}} \underline{v}\overline{Q}(\underline{m} + \Sigma m_{-i}). \end{aligned}$$

As a result, $m_i = \underline{m}$ guarantees a non-negative ex post payoff. An interpretation is that $\overline{\mathcal{M}}(\underline{m}, K)$ is a *discrete proportional auction*, where allocation and transfers are proportional, and in addition, every bidder receives a constant subsidy of $L_p\underline{m}$.

[32] The Lipschitz continuity of $\overline{T}^p = \overline{T} - \underline{v}\overline{Q}$ is established in the proof of Lemma 14.

Proposition 5. *For all $\epsilon > 0$, there exist $\underline{m}$ and K such that for any information structure $\mathcal{S}$ and equilibrium β of $(\overline{\mathcal{M}}(\underline{m}, K), \mathcal{S})$, $\Pi(\overline{\mathcal{M}}(\underline{m}, K), \mathcal{S}, \beta) \geq \overline{\Pi} - \epsilon$.*

Next, given a positive integer K, let $\overline{S}(K)$ be the information structure derived from $\overline{S}$ by coarsening each bidder's information, so that instead of observing s_i, bidder i only observes the element of the following partition that contains s_i:

$$\left\{[0, K^{-1}), [K^{-1}, 2K^{-1}), \ldots, [K - K^{-1}, K), [K, \infty)\right\}.$$

Proposition 6. *For all $\epsilon > 0$, there exists a K such that for every mechanism $\mathcal{M}$ and equilibrium β of $(\mathcal{M}, \overline{\mathcal{S}}(K))$, $\Pi(\mathcal{M}, \overline{\mathcal{S}}(K), \beta) \leq \overline{\Pi} + \epsilon$.*

Thus, the solution we constructed is finitely approximable, and the finite approximations are natural discrete analogues of their counterparts in the strong maxmin solution. The proofs of these propositions follow the arguments in Section 4, adapted to the discrete setting. We note that analogous statements of Theorem 3 and other results of this section hold for the must-sell case. This is discussed further in Appendix B.

We now present our uniqueness result:

Theorem 4 (Uniqueness)**.** *Every finitely approximable strong maxmin solution has a profit guarantee of $\overline{\Pi}$.*

Proof of Theorem 4. If a strong maxmin solution has value Π and is finitely approximable, then for any $\epsilon > 0$, there exists $\mathcal{M}^F \in \mathbf{M}^F$ such that expected profit in any information structure and equilibrium is at least $\Pi - \epsilon$. In particular, for any K, there exists an equilibrium of $(\mathcal{M}^F, \overline{\mathcal{S}}(K))$ (since both are finite) in which profit is at least $\Pi - \epsilon$. By Proposition 6, there is a K such that profit in this equilibrium is at most $\overline{\Pi} + \epsilon$. This shows that $\Pi - \epsilon \leq \overline{\Pi} + \epsilon$. As ϵ was arbitrary, we conclude that $\Pi \leq \overline{\Pi}$. The reverse inequality follows from an analogous argument, using a finite approximation $\mathcal{S}^F$ and $\overline{\mathcal{M}}(\underline{m}, K)$. □

An additional implication of Theorem 3 is that the finite approximations of $\overline{\mathcal{M}}$ and $\overline{\mathcal{S}}$ are solutions of a large family of maxmin mechanism design and minmax information design problems:

Corollary 1. *Fix an arbitrary selection $\beta^*(\mathcal{M}, \mathcal{S})$ from the (non-empty) equilibrium correspondence B on $\mathbf{M}^F \times \mathbf{S}^F$. Then*

$$\sup_{\mathcal{M} \in \mathbf{M}^F} \inf_{\mathcal{S} \in \mathbf{S}^F} \Pi(\mathcal{M}, \mathcal{S}, \beta^*(\mathcal{M}, \mathcal{S})) = \inf_{\mathcal{S} \in \mathbf{S}^F} \sup_{\mathcal{M} \in \mathbf{M}^F} \Pi(\mathcal{M}, \mathcal{S}, \beta^*(\mathcal{M}, \mathcal{S})) = \overline{\Pi}.$$

Moreover, the sup in the first term of the above equation is attained by the discrete proportional auctions $\overline{\mathcal{M}}(\underline{m}, K)$, and the inf in the second term is attained by the information structures $\overline{\mathcal{S}}(K)$.

This result provides an additional foundation for proportional auctions, as limits of ϵ-equilibria of zero-sum games where we fix the equilibrium selection rule.

Corollary 1 can be strengthened in the following manner. The reason for appealing to finite approximations is to ensure existence of an equilibrium for a suitably large class of

alternative mechanisms or information structures. Let us say that an information structure $\mathcal{S}$ is *regular* if for all $\mathcal{M} \in \mathbf{M}^F$, the game $(\mathcal{M}, \mathcal{S})$ has an equilibrium. A mechanism $\mathcal{M}$ is *regular* if for all $\mathcal{S} \in \mathbf{S}^F$, the game $(\mathcal{M}, \mathcal{S})$ has an equilibrium.

Due to the non-compactness of the signal space, we do not know whether $\overline{\mathcal{S}}$ is regular. It is, however, easy to regularize $\overline{\mathcal{S}}$ by adding an infinite signal and defining $\overline{w}(\infty) = \overline{v}$, so that $\overline{w}$ is continuous at infinity. The extended information structure $\overline{\mathcal{S}}^*$ is regular, since for any finite mechanism, the associated Bayesian game satisfies the sufficient conditions in Milgrom and Weber (1985). One can also extend the arguments of Section 4.2 to show that equilibrium profit is at most $\overline{\Pi}$. As a result, if the domain of the infimum in Corollary 1 is a set of regular information structures that contains $\overline{\mathcal{S}}^*$, then the inf is attained by $\overline{\mathcal{S}}^*$.

It is an open question whether or not $\overline{\mathcal{M}}$ can be regularized by a similar technique.[33] The difficulty is how to define the allocation and transfer at infinity. There are, however, other mechanisms that are regular and have the same profit lower bound. In particular, an earlier version of this paper constructed a maxmin mechanism with the same allocation and the following transfer:

$$\bar{t}_i(m) = \underline{v}\overline{q}_i(m) + \frac{1}{N!}\sum_{\zeta \in Z} \int_{x=0}^{\infty} \left(\overline{\Xi}^p(\Sigma m_{\zeta<\zeta(i)} + x) - \overline{\Xi}^p(\Sigma m_{\zeta\leq\zeta(i)} + x)\right) g_{N-\zeta(i)+1}(x)\, dx,$$

where $\overline{\Xi}^p(x) = \overline{\Xi}(x) - \underline{v}(\overline{\mu}(x) - \overline{Q}(x))$, Z is the set of permutations of $\{1, \ldots, N\}$, $m_{\zeta\leq k}$ is the subvector of messages $m_{\{j|\zeta(j)\leq k\}}$, and $m_{\zeta<k}$ is the subvector $m_{\{j|\zeta(j)<k\}}$. This transfer rule is continuous at infinity. If the allocation is extended so that the good is equally shared between bidders who submit infinite bids and transfers are extended via continuity, then the resulting mechanism is regular. Indeed, with a finite information structure, the resulting Bayesian game is payoff secure and upper semi-continuous, so that existence of an equilibrium follows from Reny (1999). More details can be found in the working paper Brooks and Du (2019).

In addition to the solutions we have described, there may be other solutions to the allocation sensitivity and excess growth equations. Indeed, when the support of H is $\{0, 1\}$, distinct allocation and transfer rules are constructed by Bergemann, Brooks, and Morris (2016). As discussed in Section 3, however, the allocation rule is unique in the must-sell case when $N = 2$ and the support of H is convex, and when we restrict attention to continuous allocation rules and one-dimensional bids. There may also be more exotic solutions, such as mechanisms that explicitly elicit belief hierarchies. The characterization of the set of strong maxmin solutions is an interesting topic for future work.

[33] Proposition 5 shows that it is possible to approximate $\overline{\mathcal{M}}$ with finite mechanisms, which are necessarily regular. In fact, it can be shown that the variant of $\overline{\mathcal{M}}$ where bids are capped at some $\overline{m} > 0$ is regular, and similar steps as in the proof of Proposition 5 can be used to show that it has a profit lower bound that converges to $\overline{\Pi}$ as $\overline{m} \to \infty$.

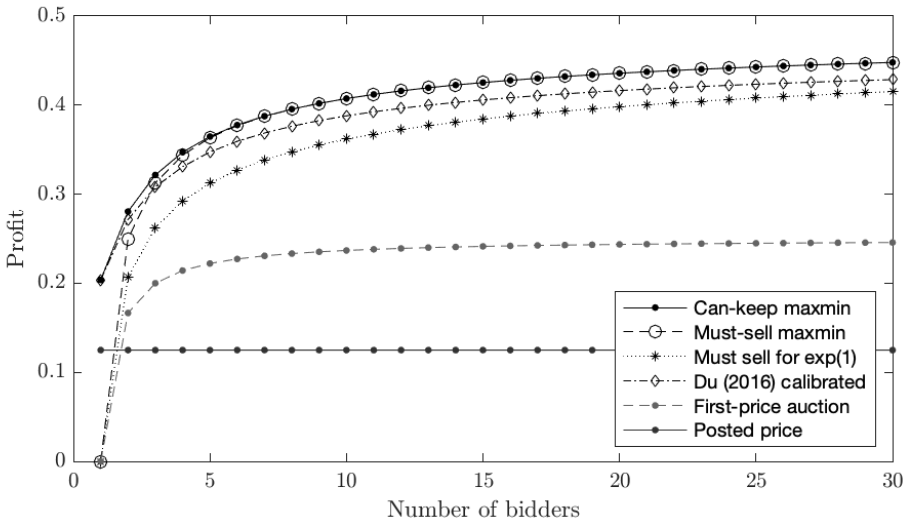

Figure 4: Comparing maxmin mechanisms to other mechanisms.

6 Maxmin auctions in the many-bidder limit

6.1 Profit comparison

In this section, we further explore the properties of the maxmin proportional auctions and the optimal profit guarantee. We begin with a comparison of mechanisms for the standard uniform distribution with $c = 0$, for which the optimal profit guarantee was computed in Section 4.4. In Figure 4, we have plotted the optimal guarantee for N ranging from 1 to 30.[34] The can-keep and must-sell guarantees are the dots and circles, respectively.

For comparison, the gray dots are the profit guarantee of the first-price auction, as computed by Bergemann, Brooks, and Morris (2017), which is $(N-1)/(4N-2)$. Also, the solid black line the best guarantee from a posted price mechanism, which is 1/8 and is obtained with a price of 1/4.[35]

A striking feature of this picture is that the optimal profit guarantee increases in N and appears to be converging towards 0.5. The latter is the ex ante expected value, which is obviously an upper bound on profit in any mechanism. In fact, as N goes to infinity, the optimal profit guarantee converges to the expected surplus. This remarkable fact is implied by the earlier result of Du (2018), who constructed a particular sequence of mechanisms and profit guarantees (the diamonds) which converge to the expected value. A fortiori, the optimal profit guarantee must as well.

The rest of this section explores and extends this result. We generalize the bound to positive production costs, in which case the correct limit profit guarantee is the ex ante gains from trade. The optimal rate of convergence is characterized. We also show that the limit is attained even with must-sell mechanisms. And perhaps most surprisingly, we argue that the same limit holds even if the distribution of the value is incorrectly specified. As

[34] A similar figure previously appeared in Du (2018).

[35] The worst-case information is a public signal indicating whether the value is above or below 1/2.

an illustration, the triangles in Figure 4 are a profit guarantee for the maxmin auction that is calibrated to an exponentially distributed value but when the value is actually standard uniform.[36] Finally, we describe the limiting maxmin aggregate allocation and transfer.

6.2 Information and welfare in the many-bidder limit

We now proceed formally. As a preliminary step, we address the left-tail assumption on the value distribution introduced in Section 2, which were only assumed for a single N, whereas we now take N to infinity. It turns out, however, that no additional assumption is needed:

Lemma 9. *If the left-tail assumption holds for N, it also holds for any $N' > N$.*

Proofs for all results of this section are in Appendix C.

We now denote the optimal profit guarantees for the can-keep and must-sell models by $\overline{\Pi}_N(H)$ and $\widehat{\Pi}_N(H)$, respectively, emphasizing their dependence on the number of bidders and the distribution. The production cost is held fixed.

A simple upper bound on the profit guarantee that holds for all N is the ex ante gains from trade. For if the bidders have no information about the value, the best the Seller can do is make a take-it-or-leave-it offer at a price equal to the ex ante expected value. We now show that this upper bound is tight:

Proposition 7 (Limiting profit guarantee)**.** *In the limit as N goes to infinity, the profit guarantees $\overline{\Pi}_N(H)$ and $\widehat{\Pi}_N(H)$ converge to the ex ante gains from trade at a rate of $1/\sqrt{N}$.*

Here is a sketch of the argument. Recall that under the minmax information, the aggregate signal is a sufficient statistic for the value. Let us change the units of each bidder's signal according to[37] $s_i^C = (s_i - 1)/\sqrt{N}$, where the "C" denotes a central limit normalization. The centered aggregate signal $x = \Sigma s^C = (\Sigma s - N)/\sqrt{N}$ has cumulative distribution $G_N^C(x) = G_N(\sqrt{N}x + N)$ and density $g_N^C(x) = \sqrt{N} g_N(\sqrt{N}x + N)$, respectively. We can correspondingly center the value function as $\overline{w}_N^C(x) = \overline{w}_N(\sqrt{N}x + N)$, etc, where we now emphasize the dependence of $\overline{w}$ and other objects on N.

The Lindeberg-Lévy theorem implies that the distribution of the centered aggregate signal converges to a standard Normal with distribution Φ and density ϕ. We argue in Appendix C that the normalized fully-revealing gains function converges almost surely to $\widehat{\gamma}_\infty^C(x) = H^{-1}(\Phi(x)) - c$, which is just a change of units from $\widehat{\gamma}_N$, and the graded gains function converges almost surely to

$$\overline{\gamma}_\infty^C(x) = \begin{cases} 0 & \text{if } x < x^*; \\ H^{-1}(\Phi(x)) - c & \text{if } x \geq x^*, \end{cases}$$

[36] In Section 2, we assumed that the support of H is bounded, which is violated by the exponential distribution. In this calculation, we have taken the limit of the formulae for bounded distributions. We suspect that our results extend to unbounded distributions as long as the right tail is not too heavy.

[37] The discussion here uses the standard central limit normalization. In Appendix C, we use a different but asymptotically equivalent normalization: $s_i^C = (s_i - (N-1)/N)/\sqrt{N-1}$. This turns out to be much more analytically convenient, e.g., in the proof of Lemma 28.

where x^* is the largest x such that

$$0 = \int_{y=-\infty}^{x} \widehat{\gamma}_{\infty}^{C}(y)\phi(y)dy. \tag{29}$$

(Note that x^* is $-\infty$ if $v - c > 0$ with probability one.) Thus, in the limit, there is only grading at the bottom, and then only if the gains from trade may be negative.

With this normalization, the hazard rate of each bidder's signal becomes $\sqrt{N}$, so that when N is large, each bidder's virtual value is approximately

$$\overline{\gamma}_{\infty}^{C}(x) - \frac{1}{\sqrt{N}}\frac{d}{dx}\overline{\gamma}_{\infty}^{C}(x).$$

Thus, information rents go to zero at a rate of $1/\sqrt{N}$. Since only the bidder who is allocated the good gets an information rent, we conclude that total bidder surplus goes to zero at a rate of $1/\sqrt{N}$. At the same time, it is always weakly optimal for the Seller to allocate the good, so that profit converges to the ex ante gains from trade.

This sketch glosses over significant technical complications. The convergence of the gains function is only almost everywhere, and along the sequence of minmax information structures, the hazard rate and the graded gains function are both changing. The formal proof deals with these issues by working directly with the integral for the difference between ex ante gains from trade and profit, scaled up by $\sqrt{N}$. This sequence converges to a positive constant, thus establishing the proposition.

6.3 Robustness to the prior

We have assumed that the Seller does not know the information structure but knows the value distribution exactly. There is a clear tension here. It turns out, however, that our results are robust to misspecification of the prior, as we now explain.

Suppose that the Seller runs the maxmin proportional auction for the prior H, denoted $\overline{\mathcal{M}}_N(H)$. Let $\overline{\lambda}_N(v; H)$ denote the associated optimal multipliers given by (18). The proof of Proposition 3 establishes that a lower bound on profit is the expectation of $\overline{\lambda}_N(v; H)$. In that argument, the prior H only appears at the last step as a mean-preserving spread of the distribution of $w(s)$. As a result, even if the prior is some $H' \neq H$, we still obtain a lower bound on profit, which is the expectation of $\overline{\lambda}_N(v; H)$ under H'. Since $\overline{\lambda}_N(v; H)$ is bounded and continuous, the change in the profit guarantee is small as long as H is close to H' in the weak-* topology.

Proposition 8 (Profit guarantee for misspecified prior)**.** *Fix a distribution H' with support contained in $[\underline{v}, \overline{v}]$. In any equilibrium of $\overline{\mathcal{M}}_N(H)$ for any information structure where the value distribution is H', expected profit is at least*

$$\overline{\Pi}_N(H, H') = \int_{v=\underline{v}}^{\overline{v}} \overline{\lambda}_N(v; H)H'(dv),$$

which is a linear and weak- continuous function of H'.*

Thus, when the prior is only slightly misspecified, the loss in the profit guarantee is small. If the prior is badly misspecified, the loss may be substantial. But when the number of bidders is large, the loss from misspecification vanishes:

Proposition 9 (Prior-independent limiting profit guarantee)**.** *Fix a distribution H' with support contained in $[\underline{v}, \overline{v}]$. As N goes to infinity, $\overline{\Pi}_N(H, H')$ converges to the ex ante gains from trade under H'.*

Note that this limiting profit guarantee need not be positive, in which case the optimal profit guarantee is zero and it is better to shut down production.

When $c \leq \underline{v}$, this result follows from Propositions 7 and 8. To see why, consider what would happen if the Seller ran $\overline{\mathcal{M}}_N(H)$ but the true prior puts probability one on a particular value $v \in [\underline{v}, \overline{v}]$. Proposition 8 says that profit must be at least $\overline{\lambda}_N(v; H)$. At the same time, profit in this counterfactual cannot be greater than $v - c$, which is the efficient surplus. But Proposition 7 says that expected profit guarantee under H converges to the ex ante gains from trade, which is only possible if $\overline{\lambda}_N(v; H)$ converges to $v - c$ H-almost surely.

This argument establishes Proposition 9 if H' is absolutely continuous with respect to H and there is common knowledge of (ex post) gains from trade. The result is much stronger. In Appendix C, we show that $\overline{\lambda}_N(v)$ converges pointwise to $v - c$ for all $v \in [\underline{v}, \overline{v}]$, even when there is not common knowledge of gains from trade.

Analogs of Propositions 8 and 9 also hold for the must-sell model. The necessary modifications to the proof are minor, as explained in Appendix C.

6.4 Limiting allocation and transfer

As a last topic, we present two descriptions of the limiting maxmin allocation and transfer. For this section, we focus on a relatively simple case when the value distribution is asymptotically single crossing. In particular, we assume there exists a $C > 0$ such that for all $a, b \in [\underline{v}, \overline{v}]$ such that $a \leq b$, $H(b) - H(a) \geq C(b - a)$.[38] In addition, we assume that $\underline{v} \neq c$.

Lemma 10. *Suppose that the above stated conditions hold. Then there exists an $\widehat{N}$ such that for all $N > \widehat{N}$, if $\underline{v} > c$, there are no graded intervals, and if $\underline{v} < c$, $\overline{\gamma}_N^C$ has a single graded interval of the form $[-\sqrt{N-1}, x_N]$.*

Let us next define

$$\overline{Q}_N^C(x) = \overline{Q}_N(x\sqrt{N} + N), \quad \overline{T}_N^C(x) = \overline{T}_N(x\sqrt{N} + N).$$

Proposition 10. *Under the conditions preceding Lemma 10, for all $x \in \mathbb{R}$, $\lim_{N\to\infty} \overline{Q}_N^C(x) = 1$ and*

$$\lim_{N\to\infty} \overline{T}_N^C(x) = \frac{1}{\phi(x)} \int_{y=-\infty}^{x} F(y)\phi(y)\, dy,$$

[38] This is equivalent to assuming that the absolutely continuous part of H has a density that is bounded away from zero on $[\underline{v}, \overline{v}]$.

where

$$F(x) = \begin{cases} -cx + \widehat{\gamma}^C_\infty(x^*)(1-\Phi(x^*)) + \int_{y=x^*}^\infty \widehat{\gamma}^C_\infty(y)(1-\Phi(y))\,dy & \text{if } x < x^*; \\ -cx - \widehat{\gamma}^C_\infty(x^*)\Phi(x^*) - \int_{y=x^*}^x \widehat{\gamma}^C_\infty(y)\Phi(y)\,dy + \int_{y=x}^\infty \widehat{\gamma}^C_\infty(y)(1-\Phi(y))\,dy & \text{if } x > x^*, \end{cases}$$

where $\widehat{\gamma}^C_\infty(x) = H^{-1}(\Phi(x)) - c$ *and* x^* *is the largest solution to equation* (29).

The limits for the must-sell mechanism are the same as those in Proposition 10, substituting $x^* = -\infty$.

Thus, under the central limit normalization, the good is asymptotically always allocated. One might have guessed that in this limit the aggregate transfer would be equal to the expected value under the minmax information, but Proposition 10 shows that this is not the case.

In some cases, the formula for the limit transfer simplifies substantially. The running uniform example does not satisfy the hypotheses of Lemma 10, since $\underline{v} = c = 0$. Nonetheless, in Appendix D.2, it is shown that there is no grading in the limit, and

$$\lim_{N\to\infty} \overline{T}^C_N(x) = \Phi(x) + \frac{\Phi(x) - \Phi(x\sqrt{2})}{\sqrt{\pi}\phi(x)}.$$

We get a somewhat different perspective on the limit when we use the law of large numbers normalization, in which signals are i.i.d. exponential with arrival rate N. Let us define

$$\overline{Q}^L_N(x) = \overline{Q}_N(Nx), \quad \overline{T}^L_N(x) = \overline{T}_N(Nx).$$

Proposition 11. *Under the conditions preceding Lemma 10, for all* $x \in \mathbb{R}$,

$$\lim_{N\to\infty} \overline{Q}^L_N(x) = \begin{cases} 1 & \text{if } x^* = -\infty; \\ \min\{x, 1\} & \text{otherwise}, \end{cases}$$

and

$$\lim_{N\to\infty} \overline{T}^L_N(x) = \begin{cases} \max\{\underline{v}, c\} & \text{if } x < 1; \\ \overline{v} & \text{if } x > 1, \end{cases}$$

Proof of Proposition 11. The limit of the transfer follows from Proposition 10 and the computations using L'Hôpital's rule for $\lim_{x\to-\infty}\lim_{N\to\infty}\overline{T}^C_N(x)$ and $\lim_{x\to\infty}\lim_{N\to\infty}\overline{T}^C_N(x)$.

For the allocation, if $\underline{v} < c$, then there is a finite $\widehat{N}$ such that $\overline{Q}^L_N = \min\left\{x/(x_N/\sqrt{N}+1), 1\right\}$ for all $N > \widehat{N}$. Since $x_N \to x^*$, we conclude that $\overline{Q}^L_N$ converges to $\min\{x, 1\}$. If $\underline{v} > c$, then there is a finite $\widehat{N}$ such that $\overline{Q}^L_N(x) = 1$ for all $N > \widehat{N}$. □

Thus, rationing persists in the limit under the law of large numbers normalization when the gains from trade might be negative. However, the aggregate transfer pushes the aggregate bid to 1, so that in equilibrium, the good will almost always be allocated.

7 Conclusion

This paper has studied the canonical auction design problem when values are common. The novelty is to use an informationally robust criterion for measuring auction performance. The spirit of the exercise is to identify mechanisms that are less vulnerable to misspecification of information and behavior and are therefore more viable in a practical setting, where a designer may be unwilling or unable to commit to a specific description of information.

The literature to which we contribute has previously shown that it is possible to obtain non-trivial profit guarantees across all information structures and equilibria, even with simple mechanisms like the first-price auction (Bergemann, Brooks, and Morris, 2017). It has also shown that there are mechanisms whose profit guarantees are unimprovable when the number of bidders is large (Du, 2018). Our marginal contribution is to establish, in a rich class of environments, the precise limit of what can be attained. We have also developed new methodology for the characterization of maxmin mechanisms, namely the double revelation principle and the critical conditions on the aggregate allocation sensitivity and the aggregate excess growth. Finally, we have shown that the optimal guarantee can be attained with the relatively simple class of proportional auctions, which are parametrized by just the aggregate allocation and aggregate transfer as functions of the aggregate bid. The analysis also indicates that simple aggregate allocation rules that increase linearly until the available supply is exhausted can perform well.

To our knowledge, proportional auctions are new to the literature, and we are unaware of instances where these auctions have been used in practice. We therefore view our contribution as normative. Our model stays within the Bayesian mechanism design framework, broadly defined, but also allows us to remedy conceptual and practical limitations associated with having to commit to a specific information structure. To be sure, this approach introduces new conceptual issues: Why should the bidders have common knowledge of the information structure, while the Seller does not? Why does the Seller not simply induce the bidders to reveal the information structure, and then run the optimal mechanism for the environment they report? This is clearly a theoretical possibility, but it runs contrary to our primary motivation, which is to identify mechanisms with desirable welfare properties that remain feasible when we respect the designer and the agents' limited ability to articulate higher-order beliefs. We have not imposed such constraints explicitly. To us, the value of the model is not just in its assumptions, but also in the form of the results: mechanisms that have desirable welfare properties but also feature a simple bidding interface and ruleset, so that they remain feasible in the face of additional practical constraints. As for the common prior among the agents, this is obviously a strong assumption, but we find it relatively palatable as an as-if description of agents' behavior, as long as it does not need to be explicitly communicated by the bidders and it is not a necessary input to compute the optimal mechanism.

Nonetheless, it is true that in distancing ourselves from the untenable knowledge assumptions of the standard model, we have taken an equally extreme and implausible position, which is that the designer puts no restrictions on information except for the value distribution and the existence of a common prior. Verily, the truth must lie somewhere in between. Designers may be willing to rule out some models without committing themselves

to a single description of the world. We expect the theory to become even more useful as we explore the middle ground between these two extremes, by incorporating reasonable restrictions on beliefs into the robust mechanism design problem.

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A Omitted Proofs for Section 4

Proof of Proposition 1. $(i) \implies (ii)$: Suppose $(\mathcal{M}, \mathcal{S}, \beta)$ is a strong maxmin solution with profit guarantee Π. Let X be the disjoint union of $\{M_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$. Clearly, $\mathcal{M} \in \mathbf{M}(X)$ and $\mathcal{S} \in \mathbf{S}(X)$. We will argue that $(\mathcal{M}, \mathcal{S})$ is a non-trivial Nash equilibrium of $\mathcal{G}(X, \beta^*)$ for every $\beta^* \in \mathcal{B}(X)$. Fix a selection β^*. By condition 3, $B(\mathcal{M}, \mathcal{S})$ is non-empty, so that Seller and Nature's payoffs from $(\mathcal{M}, \mathcal{S})$ are finite. Moreover, conditions 1 and 2 imply that all equilibria of $(\mathcal{M}, \mathcal{S})$ have the same profit, which is Π. The payoffs from $(\mathcal{M}, \mathcal{S})$ are therefore $(\Pi, -\Pi)$, Clearly, neither party can profit by deviating so that the resulting game has no bidder equilibria, so we restrict attention to deviations that result in an equilibrium. For any $\mathcal{M}' \in \mathbf{M}(X)$ such that $B(\mathcal{M}', \mathcal{S}) \neq \emptyset$, condition 1 implies that $\Pi(\mathcal{M}', \mathcal{S}, \beta^*(\mathcal{M}', \mathcal{S})) \leq \Pi$, and for any $\mathcal{S}' \in \mathbf{S}$ with $B(\mathcal{M}, \mathcal{S}') \neq \emptyset$, condition 1 implies that $\Pi(\mathcal{M}', \mathcal{S}, \beta^*(\mathcal{M}', \mathcal{S})) \leq \Pi$. Thus, $(\mathcal{M}, \mathcal{S})$ is a non-trivial Nash equilibrium.

$(ii) \implies (i)$: Suppose that $(\mathcal{M}, \mathcal{S}) \in \mathbf{M}(X) \times \mathbf{S}(X)$ is a non-trivial Nash equilibrium of $\mathcal{G}(X, \beta^*)$ for all $\beta^* \in \mathcal{B}(X)$. Fix $\beta \in B(\mathcal{M}, \mathcal{S})$. We claim that $(\mathcal{M}, \mathcal{S}, \beta)$ is a strong maxmin solution with profit guarantee $\Pi = \Pi(\mathcal{M}, \mathcal{S}, \beta)$. Condition 3 is immediate. We now show Condition 1. Fix any $\mathcal{M}'$ and equilibrium β' of $(\mathcal{M}', \mathcal{S})$. Then by the revelation principle, there is an incentive compatible direct mechanism on $\mathcal{S}$ for which truth-telling is an equilibrium. Since $\mathcal{M}'$ is participation-secure, bidders' payoffs must all be non-negative, so we can extend the direct mechanism by adding a participation-secure message, so that if any bidder sends the participation-secure message, the Seller keeps the good and all transfers are zero. Call this mechanism $\mathcal{M}''$, and observe that $\mathcal{M}'' \in \mathbf{M}(X)$ (since by construction $\mathbf{M}(X)$ contains all participation mechanisms with message spaces S_i plus an additional message). Since the payoff from this participation-security message is zero, the truthful strategies β'' are an equilibrium. Now, fix a selection β^* for which the truthful equilibrium β'' is selected on $(\mathcal{M}'', \mathcal{S})$ and β is selected on $(\mathcal{M}, \mathcal{S})$. Then since $(\mathcal{M}, \mathcal{S})$ is a Nash equilibrium of $\mathcal{G}(X, \beta^*)$, we conclude that

$$\begin{aligned}
\Pi(\mathcal{M}', \mathcal{S}, \beta') &= \Pi(\mathcal{M}'', \mathcal{S}, \beta'') \\
&= \Pi(\mathcal{M}'', \mathcal{S}, \beta^*(\mathcal{M}'', \mathcal{S})) \\
&\leq \Pi(\mathcal{M}, \mathcal{S}, \beta^*(\mathcal{M}, \mathcal{S})) \\
&= \Pi(\mathcal{M}, \mathcal{S}, \beta) = \Pi.
\end{aligned}$$

This proves condition 1. Condition 2 follows by an analogous argument, where we use the well-known fact that for any $\mathcal{S}'$ and equilibrium of $(\mathcal{M}, \mathcal{S}')$, there is a Bayes correlated equilibrium $\mathcal{S}''$ that, together with the obedient strategies, induces the same profit. Moreover, this BCE, using the message space of $\mathcal{M}$ as the signal space, is necessarily in $\mathbf{S}(X)$. □

Lemma 11. *The ex ante expectation of $\overline{\Xi}$ is zero:*

$$\int_{x=0}^{\infty} \overline{\Xi}(x) g_N(x)\, dx = 0.$$

Proof of Lemma 11. Using the formula for $\overline{\Xi}$ in equation (19) and Lemma 6, it is sufficient to show that

$$\overline{\Pi} = \int_{x=0}^{\infty} \left[\overline{\mu}(x)\widehat{w}(x) - c\overline{Q}(x)\right] g_N(x)\, dx. \tag{30}$$

From (22), note that

$$\begin{aligned}\int_{x=0}^{\infty}\left(\overline{\mu}(x)-\overline{Q}(x)\right)g_N(x)dx &= \int_{x=0}^{\infty}\left(\overline{Q}'(x)+\left(\frac{N-1}{x}-1\right)\overline{Q}(x)\right)g_N(x)dx \\ &= \int_{x=0}^{\infty}\left(\overline{Q}'(x)g_N(x)+\left(g_{N-1}(x)-g_N(x)\right)\overline{Q}(x)\right)dx \qquad (31)\\ &= \int_{x=0}^{\infty}\frac{d}{dx}\left(g_N(x)\overline{Q}(x)\right)dx = 0,\end{aligned}$$

since $g_N(0)\overline{Q}(0)=0$ and $g_N(x)\overline{Q}(x)$ goes to zero as $x\to\infty$. Thus, (30) is equivalent to

$$\overline{\Pi}=\int_{x=0}^{\infty}(\widehat{w}(x)-c)\overline{\mu}(x)g_N(x)dx.$$

Using $\widehat{\gamma}(x)=\widehat{w}(x)-c$ and the formula for $\overline{\Pi}$ in (3), we see that the above equation is equivalent to

$$\int_{x=0}^{\infty}\overline{\gamma}(x)g_{N-1}(x)dx=\int_{x=0}^{\infty}\widehat{\gamma}(x)\overline{\mu}(x)g_N(x)dx. \qquad (32)$$

When $\overline{\Gamma}(x)=\widehat{\Gamma}(x)$, we have $\overline{\gamma}(x)=\widehat{\gamma}(x)$ and $\overline{\mu}(x)=(N-1)/x$, so $\overline{\mu}(x)g_N(x)=g_{N-1}(x)$ and the two integrands are exactly equal. On the other hand, over a graded interval $[a,b]$,

$$\begin{aligned}\int_{x=a}^{b}\widehat{\gamma}(x)\overline{\mu}(x)g_N(x)dx &= C(a,b)(\widehat{\Gamma}(b)-\widehat{\Gamma}(a))\\ &= C(a,b)(\overline{\Gamma}(b)-\overline{\Gamma}(a))\\ &= C(a,b)\int_{x=a}^{b}\overline{\gamma}(x)g_N(x)dx\\ &= \int_{x=a}^{b}\overline{\gamma}(x)g_{N-1}(x)dx,\end{aligned}$$

where the equality follows from $\overline{\gamma}(x)=\overline{\gamma}(a)\exp(x-a)$ and for $n\geq 1$,

$$\int_{x=a}^{b}\overline{\gamma}(x)g_n(x)dx=\overline{\gamma}(a)\exp(-a)\frac{x^n}{n!}\bigg|_{x=a}^{b}=\overline{\gamma}(a)\exp(-a)\frac{b^n-a^n}{n!}.$$

□

Lemma 12. *For all $x>0$, $|\overline{Q}'(x)|\leq(N-1)/x$ and if $\Sigma m=x$, then*

$$\left|\frac{1}{\Delta}\left(\overline{q}_i(m_i+\Delta,m_{-i})-\overline{q}_i(m)\right)\right|\leq\frac{N+1}{x}.$$

Proof of Lemma 12. Note that on a graded interval,

$$\overline{Q}'(x)=\frac{C(a,b)}{N}-\frac{(N-1)D(a,b)}{x^N}.$$

This is at most $1/x$ when we replace $C(a,b)$ with the bound in equation (21) and set $D(a,b)=0$. Moreover, since $C(a,b)\geq 0$ and

$$D(a,b)=\frac{b^N-ab^{N-1}}{b^N-a^N}a^{N-1}\leq a^{N-1}\leq x^{N-1}, \tag{33}$$

we conclude that $\overline{Q}'$ is at least $-(N-1)/x$, so $|\overline{Q}'(x)|\leq (N-1)/x$.

Next, observe that

$$\begin{aligned}
&\left|\frac{1}{\Delta}\left(\overline{q}_i(m_i+\Delta,m_{-i})-\overline{q}_i(m)\right)\right| \\
&=\left|\frac{1}{\Delta}\left(\frac{m_i+\Delta}{\Sigma m+\Delta}\overline{Q}(\Sigma m+\Delta)-\frac{m_i}{\Sigma m}\overline{Q}(m)\right)\right| \\
&=\left|\frac{1}{\Delta}\left(\frac{m_i}{\Sigma m+\Delta}(\overline{Q}(\Sigma m+\Delta)-\overline{Q}(\Sigma m))+\frac{\Delta}{\Sigma m+\Delta}\overline{Q}(\Sigma m+\Delta)+\left(\frac{m_i}{\Sigma m+\Delta}-\frac{m_i}{\Sigma m}\right)\overline{Q}(m)\right)\right| \\
&\leq\left|\frac{m_i}{\Delta}\frac{\overline{Q}(\Sigma m+\Delta)-\overline{Q}(m)}{\Sigma m+\Delta}\right|+\left|\frac{\overline{Q}(\Sigma m+\Delta)}{\Sigma m+\Delta}\right|+\left|\frac{m_i}{\Sigma m(\Sigma m+\Delta)}\overline{Q}(\Sigma m)\right| \\
&\leq\frac{m_i}{\Delta}\frac{\Delta\frac{N-1}{x}}{\Sigma m+\Delta}+\frac{2}{\Sigma m}\leq\frac{N+1}{x}.
\end{aligned}$$

where the last line follows from the facts that $\overline{Q}'(x)\leq (N-1)/x$ and $\overline{Q}(x)\leq 1$. □

Lemma 13. $\lim_{x\to\infty}\overline{Q}(x)=1$.

Proof of Lemma 13. For each $x\in\mathbb{R}_+$, either x is not graded, in which case $\overline{Q}(x)=1$, or x is in a graded interval $[a,b]$, in which case $|\overline{Q}(x)-1|\leq\frac{N-1}{a}(b-a)$ since $\overline{Q}(a)=1$ and $|\overline{Q}'(x)|\leq\frac{N-1}{a}$ by Lemma 12. But the length of a graded interval $[a,b]$ is bounded above by a constant $C=\log(\overline{v})-\log(\overline{\gamma}(0))$ since

$$\overline{v}\geq\overline{\gamma}(b)=\overline{\gamma}(a)\exp(b-a)\geq\overline{\gamma}(0)\exp(b-a).$$

So $|\overline{Q}(x)-1|\leq\frac{N-1}{x-C}C$ when x is in a graded interval $[a,b]$. In any case, we conclude that $\lim_{x\to\infty}\overline{Q}(x)=1$. □

For proving Lemma 8, it is convenient to decompose $\overline{t}_i(m)$ into a "base" component of $\underline{v}\,\overline{q}_i(m)$ (whose derivative may be unbounded around 0) and a "premium" component of $\overline{t}_i^p(m)=\overline{t}_i(m)-\underline{v}\,\overline{q}_i(m)$ (whose derivative is bounded; see Lemma 14 below). Let $\overline{\Xi}^p(x)=\overline{\Xi}(x)-\underline{v}(\overline{\mu}(x)-\overline{Q}(x))$, and

$$\begin{aligned}
\overline{T}^p(x)=\overline{T}(x)-\underline{v}\overline{Q}(x)&=\frac{1}{g_N(x)}\left(\int_{y=0}^{x}\overline{\Xi}(y)g_N(y)\,dy-\underline{v}\overline{Q}(x)g_N(x)\right) \\
&=\frac{1}{g_N(x)}\int_{y=0}^{x}\overline{\Xi}^p(y)g_N(y)\,dy,
\end{aligned}$$

where the last equality follows from $\frac{d}{dx}\left(\overline{Q}(x)g_N(x)\right) = (\overline{\mu}(x)-\overline{Q}(x))g_N(x)$ by the reasoning in (31). Since both $\overline{Q}$ and $\overline{T}$ are right-differentiable (Lemmas 5 and 7), so is $\overline{T}^p$, and it is easy to verify the differential equation

$$\left(\frac{N-1}{x}-1\right)\overline{T}^p(x)+\frac{d}{dx}\overline{T}^p(x)=\overline{\Xi}^p(x). \tag{34}$$

Lemma 14. *The ratio*

$$\frac{\overline{t}_i^p(m_i+\Delta,m_{-i})-\overline{t}_i^p(m)}{\Delta}$$

is bounded over all $m \in \overline{M}$ and $\Delta > 0$.

Proof of Lemma 14. First, we show that $\overline{\Xi}^p$ is bounded. Given that $\widehat{w}(x)$ is bounded above and $\overline{\mu}(x)$ is decreasing from Lemma 3, it suffices to show that $\limsup_{x\to 0}(\widehat{w}(x)-\underline{v})/x < \infty$. This is a direct implication of part one of the left-tail assumption, since $x^{\varphi} < x$ for x sufficiently small.

Next, suppose $\Sigma m > 0$. Since $\overline{t}_i^p(m)$ is right-continuous at m, we have

$$\left|\frac{\partial \overline{t}_i^p(m)}{\partial m_i}\right| = \left|\frac{\Sigma m_{-i}}{(\Sigma m)^2}\overline{T}^p(\Sigma m)+\frac{m_i}{\Sigma m}\frac{d\overline{T}^p}{dx}(\Sigma m)\right| \le \left|\frac{\overline{T}^p(\Sigma m)}{\Sigma m}\right| + \left|\frac{d\overline{T}^p}{dx}(\Sigma m)\right|$$

We argue that the right-hand side of the above equation is bounded over all $m \neq 0$, which implies the boundedness of $(\overline{t}_i^p(m_i+\Delta,m_{-i})-\overline{t}_i^p(m))/\Delta$. Since $\overline{T}^p$ satisfies equation (34) and $\overline{\Xi}^p$ and $\overline{T}^p$ are bounded, it suffices to show that $\overline{T}^p(x)/x$ is bounded. This follows from L'Hôpital's rule:

$$\begin{aligned}
\lim_{x\to 0}\frac{\overline{T}^p(x)}{x} &= \lim_{x\to 0}\frac{\int_{y=0}^{x}\overline{\Xi}^p(y)g_N(y)\,dy}{xg_N(x)} \\
&= \lim_{x\to 0}\frac{\overline{\Xi}^p(x)g_N(x)}{g_N(x)+x(g_{N-1}(x)-g_N(x))} \\
&= \lim_{x\to 0}\frac{\overline{\Xi}^p(x)}{1+x((N-1)/x-1)} \\
&= \frac{-\overline{\lambda}(\underline{v})+(\underline{v}-c)\overline{Q}(0)}{N}.
\end{aligned} \tag{35}$$

Finally, for $m = 0$, we note $(\overline{t}_i^p(\Delta,0)-\overline{t}_i^p(0))/\Delta = \overline{T}^p(\Delta)/\Delta$ and again appeal to the boundedness of $\overline{T}^p(x)/x$. □

Lemma 15. *There exists $b > 0$ such that either $[0,b]$ is a graded interval, or $[0,b]$ is a non-graded interval.*

Proof of Lemma 15. Case 1: $\underline{v} \le c$. If there were no graded interval at zero, then we would have $\overline{\gamma}(0) = \widehat{\gamma}(0) = \underline{v} - c \le 0$. Lemma 2 then implies that for all $x \ge 0$, $\overline{\gamma}(x) \le \overline{\gamma}(0)\exp(x) \le 0$, so that $\overline{\gamma}$ has a non-positive expectation. This contradicts Lemma 1, which implies that the expectation of $\overline{\gamma}$ equals the expectation of $v - c$ under H, and the hypothesis that the latter expectation is strictly positive.

Case 2: $\underline{v} > c$. Part two of the left-tail assumption says there exists an $\epsilon > 0$ such that

$$\frac{\widehat{\gamma}(x')}{\exp(x')} \geq \frac{\widehat{\gamma}(x)}{\exp(x)}$$

for all $0 \leq x' \leq x \leq \epsilon$. Thus $\widehat{\Gamma} \circ E^{-1}$ is concave on the interval $[0, E(\epsilon)]$, so that if a subset $[a, b]$ of $[0, E(\epsilon)]$ is not graded, then $[0, a]$ is not graded as well. The claim of the lemma follows since either $[0, E(\epsilon)]$ is a subset of a graded interval, or there exists a non-graded interval that starts at zero. □

Proof of Lemma 8. Fix an information structure $\mathcal{S}$ and an equilibrium β of $(\overline{\mathcal{M}}, \mathcal{S})$. Let us rewrite equation (25) as

$$\int_S \int_{\overline{M}} \left[(w(s) - \underline{v}) \nabla \cdot \overline{q}(m) - \nabla \cdot \overline{t}^p(m) \right] \beta(dm|s)\pi(ds) \leq 0.$$

Equation (35) implies that $\nabla \cdot \overline{t}^p(0) = -\overline{\lambda}(\underline{v}) + (\underline{v} - c)\overline{Q}(0)$.

Since $(\overline{t}_i^p(m_i + \Delta, m_{-i}) - \overline{t}_i^p(m))/\Delta$ is bounded (Lemma 14), the dominated convergence theorem then implies that

$$\lim_{\Delta \searrow 0} \int_S \int_M \frac{\overline{t}_i^p(m_i + \Delta, m_{-i}) - \overline{t}_i^p(m)}{\Delta} \beta(dm|s)\pi(ds) = \int_S \int_M \frac{\partial \overline{t}_i^p}{\partial m_i}(m) \beta(dm|s)\pi(ds). \quad (36)$$

We want to argue that

$$\begin{aligned} &\liminf_{\Delta \searrow 0} \int_S \int_M (w(s) - \underline{v}) \frac{\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m)}{\Delta} \beta(dm|s)\pi(ds) \\ &\geq \int_S \int_M (w(s) - \underline{v}) \frac{\partial \overline{q}_i}{\partial m_i}(m) \beta(dm|s)\pi(ds) \end{aligned} \quad (37)$$

by appealing to Fatou's lemma: notice that $\frac{\partial \overline{q}_i}{\partial m_i}(m) = \lim_{\Delta \searrow 0} (\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m))/\Delta$ is well defined when $m \neq 0$ by Lemma 5; for $m = 0$, the limit is $\frac{\partial \overline{q}_i}{\partial m_i}(m) = \frac{1}{b}$ if $[0, b]$ is a graded interval and $\frac{\partial \overline{q}_i}{\partial m_i}(m) = \infty$ otherwise.

We consider two cases. Case 1: If there is no grading anywhere (so $\overline{q}_i(m) = m_i/\Sigma m$), or if $[0, b]$ is the only graded interval for some $b > 0$ (so $\overline{q}_i(m) = m_i/\max(\Sigma m, b)$), then Fatou's lemma applies to (37) since $\overline{q}_i(m_i + \Delta, m_{-i}) \geq \overline{q}_i(m)$ for all m and all $\Delta > 0$.

Case 2: Let $\underline{a}$ be the infimum of left end points of graded intervals that are strictly positive (of which there must be at least one if we are not in Case 1); Lemma 15 implies that $\underline{a} > 0$. We claim

$$\frac{\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m)}{\Delta} \geq -\frac{(N+1)}{\underline{a}}. \quad (38)$$

holds for all m and all $\Delta > 0$, so Fatou's lemma also applies to (37) in this case.

If $\Sigma m \geq \underline{a}$, then (38) follows from Lemma 12. If $\Sigma m + \Delta \leq \underline{a}$, then $\overline{q}_i(m_i + \Delta, m_{-i}) \geq \overline{q}_i(m)$ by examining the functional form of $\overline{q}$ as in Case 1, so (38) again holds.

Now suppose $\Sigma m < \underline{a}$ and $\Sigma m + \Delta > \underline{a}$. let $m_i' = \underline{a} - \Sigma m_{-i}$. Clearly, we have

$$\frac{\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m)}{\Delta} = \delta \frac{\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m_i', m_{-i})}{m_i + \Delta - m_i'} + (1 - \delta) \frac{\overline{q}_i(m_i', m_{-i}) - \overline{q}_i(m)}{m_i' - m_i}$$

for $\delta = (m_i + \Delta - m_i')/\Delta \in (0, 1)$. By applying Lemma 12 to the term following δ and $\overline{q}_i(m_i', m_{-i}) \geq \overline{q}_i(m)$ to the term following $1 - \delta$, we conclude that (38) again holds.

Since β is an equilibrium, for any bidder i and any fixed $\Delta > 0$ we have

$$\int_S \int_M \left[(w(s) - \underline{v}) \frac{\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m)}{\Delta} - \frac{\overline{t}_i^p(m_i + \Delta, m_{-i}) - \overline{t}_i^p(m)}{\Delta} \right] \beta(dm|s)\pi(ds) \leq 0.$$

Therefore, equations (36) and (37) imply (25) if we sum across i. □

Lemma 16. *There exists a $\widehat{\Delta}$ such that for all $\Delta < \widehat{\Delta}$ and $x \in \mathbb{R}_+$,*

$$\overline{\gamma}(x) \left[g_N(x) + \frac{g_N(x) - g_N(x - \Delta)}{\exp(\Delta) - 1} \right] \leq \begin{cases} \overline{\gamma}(x)(g_N(x) + 2) & \text{if } x < 1; \\ \overline{\gamma}(x)[g_N(x) + 2g_{N-1}(x)] & \text{if } x \geq 1. \end{cases}$$

This bounding function is integrable.

Proof of Lemma 16. If $x \geq \Delta$,

$$\begin{aligned} \frac{g_N(x) - g_N(x - \Delta)}{\exp(\Delta) - 1} &= \frac{\exp(-x)}{(N-1)!} \frac{x^{N-1} - (x - \Delta)^{N-1} \exp(\Delta)}{\exp(\Delta) - 1} \\ &\leq \frac{\exp(-x)}{(N-1)!} \frac{x^{N-1} - (x - \Delta)^{N-1}}{\exp(\Delta) - 1} \\ &\leq \frac{\exp(-x)}{(N-1)!} (N-1) x^{N-2} \frac{\Delta}{\exp(\Delta) - 1} = g_{N-1}(x) \frac{\Delta}{\exp(\Delta) - 1}, \end{aligned}$$

where the second-to-last line follows from convexity of x^{N-1}, so $x^{N-1} - (x - \Delta)^{N-1} \leq (N-1)x^{N-2}\Delta$. Since $\Delta/(\exp(\Delta) - 1) \to 1$ as $\Delta \to 0$, we can take $\widehat{\Delta} \leq 1$ small enough so that for $\Delta < \widehat{\Delta}$, the ratio is less than 2. Also, as long as $\Delta < N - 1$, g_N is increasing for $x \in [0, \Delta]$, so that for x in this range

$$\frac{g_N(x) - g_N(x - \Delta)}{\exp(\Delta) - 1} \leq \frac{g_N(\Delta)}{\exp(\Delta) - 1}. \tag{39}$$

This expression converges to zero pointwise for $N > 2$ and converges to 1 for $N = 2$. Thus, we can take Δ small so that for $x < \Delta$, the right-hand side of (39) is less than 2.

Integrability follows from the fact that $\overline{\gamma}(x)$ is bounded by $\overline{v}$. □

B Proofs for Section 5

B.1 Proof of Proposition 5

Let $\Delta = 1/K$, and recall that the message space for $\overline{\mathcal{M}}(\underline{m}, K)$ is

$$M_i = \{\underline{m}, \underline{m} + \Delta, \ldots, \underline{m} + K\}.$$

Note that the highest message $\overline{m} = \underline{m} + K$ is at least Δ^{-1}. We shall extend the domain of the allocation and transfer rules to all of $\mathbb{R}^N_+$ for notational convenience. The discrete aggregate allocation sensitivity is

$$\mu(m) = \frac{1}{\Delta} \sum_{i=1}^{N} \mathbb{I}_{m_i < \overline{m}} (q_i(m_i + \Delta, m_{-i}) - q_i(m)),$$

and the discrete aggregate excess growth is

$$\Xi(m) = \frac{1}{\Delta} \sum_{i=1}^{N} \mathbb{I}_{m_i < \overline{m}} (t_i(m_i + \Delta, m_{-i}) - t_i(m)) - \Sigma t(m).$$

Now, define

$$\lambda(m; v) = v\mu(m) - \Xi(m) - c\overline{Q}(\Sigma m),$$

and let $\lambda(v) = \min_{m \in M} \lambda(m; v)$.

Lemma 17. *For any information structures $\mathcal{S}$ and equilibrium β of $(\mathcal{S}, \overline{\mathcal{M}}(\underline{m}, K))$, expected profit is at least $\int_V \lambda(v) H(dv)$.*

Proof of Lemma 17. The equilibrium hypothesis implies that for all i,

$$\int_S \sum_{m \in M} \big[w(s)(q_i(\min\{m_i + \Delta, \overline{m}\}, m_{-i}) - q_i(m)) \\ - (t_i(\min\{m_i + \Delta, \overline{m}\}, m_{-i}) - t_i(m)) \big] \beta(m|s)\pi(ds) \leq 0,$$

which corresponds to the incentive constraint for deviating to $\min\{m_i + \Delta, \overline{m}\}$. Summing across bidders, and dividing by Δ, we conclude that

$$\int_S \sum_{m \in M} \left[w(s)\mu(m) - \Xi(m) - \Sigma t(m) \right] \beta(m|s)\pi(ds) \leq 0.$$

Hence, expected profit is

$$\int_S \sum_{m \in M} \left[\Sigma t(m) - cQ(\Sigma m) \right] \beta(m|s)\pi(ds) \\ \geq \int_S \sum_{m \in M} \left[\Sigma t(m) - cQ(\Sigma m) + w(s)\mu(m) - \Xi(m) - \Sigma t(m) \right] \beta(m|s)\pi(ds)$$

$$
\begin{aligned}
&= \int_S \sum_{m \in M} \left[w(s)\mu(m) - \Xi(m) - cQ(\Sigma m) \right] \beta(m|s)\pi(ds) \\
&\geq \int_S \lambda(w(s))\pi(ds) \\
&\geq \int_V \lambda(v)H(dv),
\end{aligned}
$$

where the last line follows from the mean-preserving spread condition on $w(s)$ and that λ is concave, being the infimum of linear functions. □

Lemma 18. *For all $m \in M$,*

$$
\mu(m) \geq \frac{1}{\Delta} \int_{y=0}^{\Delta} \overline{\mu}(\Sigma m + y)dy - \widehat{L}(\underline{m}, \Delta),
$$

where

$$
\widehat{L}(\underline{m}, \Delta) = N(N+1)\Delta + \frac{N(N-1)}{\Delta} \left(\log(N\underline{m} + \Delta) + \frac{N\underline{m}}{N\underline{m} + \Delta} - \log(N\underline{m}) - 1 \right).
$$

Moreover, for all $\underline{m} > 0$, $\widehat{L}(\underline{m}, \Delta) \to 0$ as $\Delta \to 0$.

Proof of Lemma 18. From Lemma 12, we know that

$$
\begin{aligned}
\mu(m) &= \sum_{i=1}^{N} \frac{1}{\Delta}(q_i(m_i + \Delta, m_{-i}) - q_i(m)) - \sum_{i=1}^{N} \mathbb{I}_{m_i = \overline{m}} \frac{1}{\Delta}(q_i(m_i + \Delta, m_{-i}) - q_i(m)) \\
&\geq \sum_{i=1}^{N} \frac{1}{\Delta}(q_i(m_i + \Delta, m_{-i}) - q_i(m)) - N\frac{N+1}{\overline{m}} \\
&\geq \sum_{i=1}^{N} \frac{1}{\Delta}(q_i(m_i + \Delta, m_{-i}) - q_i(m)) - N(N+1)\Delta.
\end{aligned}
$$

Recall that

$$
\overline{\mu}(x) = \frac{N-1}{x}\overline{Q}(x) + \overline{Q}'(x).
$$

Also recall that

$$
\frac{\partial q_i(m)}{\partial m_i} = \frac{\Sigma m_{-i}}{(\Sigma m)^2}\overline{Q}(\Sigma m) + \frac{m_i}{\Sigma m}\overline{Q}'(\Sigma m).
$$

Thus,

$$
\sum_{i=1}^{N} \frac{1}{\Delta} \left(q_i(m_i + \Delta, m_{-i}) - q_i(m) \right)
$$

$$
\begin{aligned}
&= \frac{1}{\Delta}\sum_{i=1}^{N}\int_{y=0}^{\Delta}\frac{\partial q_i(m_i+y,m_{-i})}{\partial m_i}dy \\
&= \frac{1}{\Delta}\sum_{i=1}^{N}\int_{y=0}^{\Delta}\left[\frac{\Sigma m_{-i}}{(\Sigma m+y)^2}\overline{Q}(\Sigma m+y)+\frac{m_i+y}{\Sigma m+y}\overline{Q}'(\Sigma m+y)\right]dy \\
&= \frac{1}{\Delta}\int_{y=0}^{\Delta}\left[\frac{(N-1)\Sigma m}{(\Sigma m+y)^2}\overline{Q}(\Sigma m+y)+\frac{\Sigma m+Ny}{\Sigma m+y}\overline{Q}'(\Sigma m+y)\right]dy \\
&= \frac{1}{\Delta}\int_{y=0}^{\Delta}\overline{\mu}(\Sigma m+y)dy-\frac{N-1}{\Delta}\int_{y=0}^{\Delta}\frac{y}{\Sigma m+y}\left[\frac{\overline{Q}(\Sigma m+y)}{\Sigma m+y}-\overline{Q}'(\Sigma m+y)\right]dy.
\end{aligned}
$$

We need to bound the last integral from above. If x is in a non-graded interval, then $\overline{Q}(x)/x-\overline{Q}'(x)$ is just $1/x$. If x is in a graded interval $[a,b]$, then

$$
\frac{\overline{Q}(x)}{x}-\overline{Q}'(x)=\frac{C(a,b)}{N}+\frac{D(a,b)}{x^N}-\frac{C(a,b)}{N}+(N-1)\frac{D(a,b)}{x^N}=\frac{ND(a,b)}{x^N}.
$$

From equation (33), $D(a,b)\le x^{N-1}$, so that the integrand in this case is at most N/x, and

$$
\begin{aligned}
\int_{y=0}^{\Delta}\frac{y}{x+y}\left[\frac{\overline{Q}(x+y)}{x+y}-\overline{Q}'(x+y)\right]dy &\le N\int_{y=0}^{\Delta}\frac{y}{(x+y)^2}dy \\
&= N\int_{y=0}^{\Delta}\left(\frac{1}{x+y}-\frac{x}{(x+y)^2}\right)dy \\
&= N\left(\log(x+\Delta)+\frac{x}{x+\Delta}-\log(x)-1\right).
\end{aligned}
$$

The derivative with respect to x is

$$
N\left(\frac{1}{x+\Delta}-\frac{1}{x}+\frac{\Delta}{(x+\Delta)^2}\right)=N\Delta\left(\frac{1}{(x+\Delta)^2}-\frac{1}{x(x+\Delta)}\right)
$$

which is clearly negative, so subject to $x\ge N\underline{m}$, the expression is maximized with $x=N\underline{m}$, which gives us the lower bound on μ.

Moreover, as $\Delta\to 0$, $N(N+1)\Delta\to 0$, and by L'Hôpital's rule,

$$
\lim_{\Delta\to 0}\left(\frac{\log(N\underline{m}+\Delta)+\frac{N\underline{m}}{N\underline{m}+\Delta}-\log(N\underline{m})-1}{\Delta}\right)=\lim_{\Delta\to 0}\left(\frac{1}{N\underline{m}+\Delta}-\frac{N\underline{m}}{(N\underline{m}+\Delta)^2}\right)=0.
$$

□

Now let us write $\Xi^p(m)=\Xi(m)-\underline{v}(\mu(m)-Q(m))$, and recall that $\overline{\Xi}^p(x)=\overline{\Xi}(x)-\underline{v}(\overline{\mu}(x)-\overline{Q}(x))$. These are the excess growths for the "premium" transfers $t_i^p(m)=t_i(m)-\underline{v}q_i(m)$ and $\overline{t}_i^p(m)=\overline{t}_i(m)-\underline{v}\overline{q}_i(m)$, respectively. We similarly denote by $\overline{T}^p(x)=\overline{T}(x)-\underline{v}\overline{Q}(x)$ the aggregate premium transfer, and note that $\overline{T}^p$ satisfies the differential equation

$$
\left(\frac{N-1}{x}-1\right)\overline{T}^p(x)+\frac{d}{dx}\overline{T}^p(x)=\overline{\Xi}^p(x),
$$

with the boundary condition $\overline{T}^p(0)=0$.

Lemma 19. *Let L_Ξ be an upper bound on $|\overline{\Xi}^p|$ and let L_T be an upper bound on $\overline{T}^p$. Then*

$$\Xi^p(m) \leq \frac{1}{\Delta}\int_{y=0}^{\Delta} \overline{\Xi}^p(\Sigma m + y)dy + \tilde{L}(\underline{m})\frac{\Delta}{2} + NL_p\underline{m} - \frac{1}{\Delta}\sum_i \mathbb{I}_{m_i=\overline{m}}\left[\overline{t}_i^p(m_i+\Delta, m_{-i}) - \overline{t}_i^p(m)\right],$$

where

$$\tilde{L}(\underline{m}) = \left(1 + \frac{N-1}{N\underline{m}}\right)L_p + \frac{N-1}{(N\underline{m})^2}L_T.$$

Proof of Lemma 19. Recall that $\overline{T}^p$ is Lipschitz with constant L_p. Furthermore, the function $\overline{T}^p(x)(N-1)/x$ is Lipschitz on $[N\underline{m}, \infty)$, and

$$\left|\frac{d}{dx}\left(\frac{N-1}{x}\overline{T}^p(x)\right)\right| = \left|\frac{N-1}{x}\frac{d}{dx}\overline{T}^p(x) - \frac{N-1}{x^2}\overline{T}^p(x)\right| \leq \frac{N-1}{N\underline{m}}L_p + \frac{N-1}{(N\underline{m})^2}L_T = L_1(\underline{m}).$$

Using the differential equation for $\overline{T}^p$,

$$\begin{aligned}
&\frac{1}{\Delta}\int_{y=0}^{\Delta}\overline{\Xi}^p(\Sigma m + y)dy \\
&= \frac{1}{\Delta}\int_{y=0}^{\Delta}\left[\left(\frac{N-1}{\Sigma m + y} - 1\right)\overline{T}^p(\Sigma m + y) + \frac{d}{dx}\overline{T}^p(x)\Big|_{x=\Sigma m+y}\right]dy \\
&= \frac{1}{\Delta}\left[\int_{y=0}^{\Delta}\left(\frac{N-1}{\Sigma m + y} - 1\right)\overline{T}^p(\Sigma m + y)dy + \overline{T}^p(\Sigma m + \Delta) - \overline{T}^p(\Sigma m)\right] \\
&\geq \frac{1}{\Delta}\left[\int_{y=0}^{\Delta}\left(\frac{N-1}{\Sigma m + \Delta}\overline{T}^p(\Sigma m + \Delta) - L_1(\underline{m})(\Delta - y) - \overline{T}^p(\Sigma m) - L_p y\right)dy + \overline{T}^p(\Sigma m + \Delta) - \overline{T}^p(\Sigma m)\right] \\
&= \frac{1}{\Delta}\left[\Delta\frac{N-1}{\Sigma m + \Delta}\overline{T}^p(\Sigma m + \Delta) - \Delta\overline{T}^p(\Sigma m) - (L_1(\underline{m}) + L_p)\frac{\Delta^2}{2} + \overline{T}^p(\Sigma m + \Delta) - \overline{T}^p(\Sigma m)\right] \\
&= \frac{1}{\Delta}\left[\frac{\Sigma m + N\Delta}{\Sigma m + \Delta}\overline{T}^p(\Sigma m + \Delta) - \overline{T}^p(\Sigma m)\right] - \overline{T}^p(\Sigma m) - \underbrace{(L_1(\underline{m}) + L_p)}_{\equiv\tilde{L}(\underline{m})}\frac{\Delta}{2}.
\end{aligned}$$

Now, let us write $T^p(\Sigma m)$ for the aggregate transfer when the messages are m. Thus,

$$\begin{aligned}
\Xi^p(m) &= \frac{1}{\Delta}\sum_{i=1}^{N}\left[t_i^p(m_i+\Delta, m_{-i}) - t_i^p(m)\right] - T^p(\Sigma m) - \frac{1}{\Delta}\sum_{i=1}^{N}\mathbb{I}_{m_i=\overline{m}}\left[t_i^p(m_i+\Delta, m_{-i}) - t_i^p(m)\right] \\
&= \frac{1}{\Delta}\sum_{i=1}^{N}\left[\overline{t}_i^p(m_i+\Delta, m_{-i}) - \overline{t}_i^p(m)\right] - T^p(\Sigma m) - \frac{1}{\Delta}\sum_{i=1}^{N}\mathbb{I}_{m_i=\overline{m}}\left[\overline{t}_i^p(m_i+\Delta, m_{-i}) - \overline{t}_i^p(m)\right]
\end{aligned}$$

$$\leq \frac{1}{\Delta}\left[\frac{\Sigma m + N\Delta}{(\Sigma m+\Delta)}\overline{T}^p(\Sigma m+\Delta) - \overline{T}^p(\Sigma m)\right] - T^p(\Sigma m) - \frac{1}{\Delta}\sum_i \mathbb{I}_{m_i=\overline{m}}\left[\overline{t}_i^p(m_i+\Delta, m_{-i}) - \overline{t}_i^p(m)\right].$$

The lemma follows from combining these two inequalities, with the observation that $T^p(x) = \overline{T}^p(x) - NL_p\underline{m}$. □

Lemma 20. *For all $\epsilon > 0$, there exists a K such that for all m such that $\Sigma m > K$ and for all i,*

$$\frac{1}{\Delta}\left|\overline{t}_i^p(m_i+\Delta, m_{-i}) - \overline{t}_i^p(m)\right| < \epsilon.$$

Proof of Lemma 20. Since $\lim_{x\to\infty}\overline{T}^p(x) = -\overline{\Xi}^p(\infty)$, we can find a K large enough so that for $x > K$, $\left|\overline{T}^p(x) + \overline{\Xi}^p(\infty)\right| < \epsilon/4$ and $L_T/K < \epsilon/4$, and thus $\left|d\overline{T}^p(x)/dx\right| < \epsilon/2$. Thus, when $\Sigma m > K$, then using $\Delta = K^{-1}$,

$$\begin{aligned}
\frac{1}{\Delta}\left[\overline{t}_i^p(m_i+\Delta, m_{-i}) - \overline{t}_i^p(m)\right] &= \frac{1}{\Delta}\int_{y=0}^{\Delta}\frac{\partial \overline{t}_i^p(m_i+y, m_{-i})}{\partial m_i}dy \\
&= \frac{1}{\Delta}\int_{y=0}^{\Delta}\left[\frac{\Sigma m_{-i}}{(\Sigma m+y)^2}\overline{T}^p(\Sigma m+y) + \frac{m_i+y}{\Sigma m+y}\left.\frac{d}{dx}\overline{T}^p(x)\right|_{x=\Sigma m+y}\right]dy \\
&\leq \frac{L_T}{K} + \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

□

Proof of Proposition 5. We first argue that there exists $\underline{m}$ and a K such that $\lambda(m;v) \geq \inf_{m'\in\mathbb{R}^N}\overline{\lambda}(m';v) - \epsilon$ for all $m \in M$ and $v \in [\underline{v}, \overline{v}]$, where

$$\overline{\lambda}(m;v) = (v-\underline{v})\overline{\mu}(\Sigma m) - \overline{\Xi}^p(\Sigma m) + (\underline{v}-c)\overline{Q}(\Sigma m).$$

From Lemma 12, we know that $|\overline{Q}(x+y) - \overline{Q}(x)| \leq y(N-1)/\underline{m}$. Thus,

$$\begin{aligned}
\left|\overline{Q}(x) - \frac{1}{\Delta}\int_{y=0}^{\Delta}\overline{Q}(x+y)dy\right| &\leq \frac{1}{\Delta}\int_{y=0}^{\Delta}\left|\overline{Q}(x+y) - Q(x)\right|dy \\
&\leq \frac{1}{\Delta}\int_{y=0}^{\Delta} y\frac{N-1}{\underline{m}}dy = \Delta\frac{N-1}{2\underline{m}}.
\end{aligned}$$

Combining this inequality with Lemmas 18 and 19, we get that

$$\begin{aligned}
\lambda(m;v) &= (v-\underline{v})\mu(m) - \Xi^p(m) + (\underline{v}-c)\overline{Q}(\Sigma m) \\
&\geq \frac{1}{\Delta}\int_{y=0}^{\Delta}\left[(v-\underline{v})\overline{\mu}(\Sigma m+\Delta) - \overline{\Xi}^p(\Sigma m+y) + (\underline{v}-c)\overline{Q}(\Sigma m+y)\right]dy \\
&\quad - (\overline{v}-\underline{v})\widehat{L}(\underline{m},\Delta) - \overline{v}\Delta\frac{N-1}{2\underline{m}} - \frac{\Delta}{2}\tilde{L}(\underline{m}) - NL_p\underline{m}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\Delta} \sum_i \mathbb{I}_{m_i = \overline{m}} \left| \overline{t}_i^p(m_i + \Delta, m_{-i}) - \overline{t}_i^p(m) \right| \\
\geq & \inf_{\{m' | \Sigma m \leq \Sigma m' \leq \Sigma m + \Delta\}} \overline{\lambda}(m'; v) \\
&- (\overline{v} - \underline{v}) \widehat{L}(\underline{m}, \Delta) - \overline{v} \Delta \frac{N-1}{2\underline{m}} - \frac{\Delta}{2} \tilde{L}(\underline{m}) - N L_p \underline{m} \\
&- \frac{1}{\Delta} \sum_i \mathbb{I}_{m_i = \overline{m}} \left| \overline{t}_i^p(m_i + \Delta, m_{-i}) - \overline{t}_i^p(m) \right| .
\end{aligned}
$$

We can first pick $\underline{m} > 0$ so that $N L_p \underline{m} < \epsilon/2$. We can then pick K large enough (and Δ small enough) such that the remaining terms in the last two lines sum to less than $\epsilon/2$ (where for the first term in the middle line and last line, this follows from Lemmas 18 and 20, respectively). We then conclude that

$$
\lambda(m; v) \geq \inf_{m' \in \mathbb{R}_N^+} \overline{\lambda}(m'; v) - \epsilon \geq \overline{\lambda}(v) - \epsilon.
$$

Hence, $\lambda(v) \geq \overline{\lambda}(v) - \epsilon$, and Lemma 17 and Lemma 6 give the result. □

This proof goes through verbatim with the maxmin must-sell mechanism $\widehat{\mathcal{M}}$.

B.2 Proof of Proposition 6

Recall the definition of $\overline{\mathcal{S}}(K)$. Let $\Delta = 1/K$. We subsequently choose K sufficiently large (and equivalently Δ sufficiently small) to attain the desired ϵ. Note that the signal space can be written

$$
S_i = \left\{ 0, \Delta, \ldots, K^2 \Delta \right\},
$$

and the highest message is simply Δ^{-1}. The probability mass function of s_i is

$$
f_i(s_i) = \begin{cases} (1 - \exp(-\Delta)) \exp(-s_i) & \text{if } s_i < \Delta^{-1}; \\ \exp(-\Delta^{-1}) & \text{if } s_i = \Delta^{-1}. \end{cases}
$$

As a result, s_i/Δ is a censored geometric random variable with arrival rate $1 - \exp(-\Delta)$. We write $f(s) = \times_{i=1}^N f_i(s_i)$ for the joint probability, and

$$
F_i(s_i) = \sum_{s_i' \leq s_i} f_i(s_i') = \begin{cases} 1 - \exp(-s_i - \Delta) & \text{if } s_i < \Delta^{-1}; \\ 1 & \text{otherwise,} \end{cases}
$$

for the cumulative distribution. The value function is

$$
w(s) = \frac{1}{f(s)} \int_{\left\{ s' \in \mathbb{R}_+^N \middle| \tau(s_i') = s_i \forall i \right\}} \overline{w}(\Sigma s') \exp(-\Sigma s') ds',
$$

where

$$\tau(x) = \begin{cases} \Delta\lfloor x/\Delta \rfloor & \text{if } x < \Delta^{-1}; \\ \Delta^{-1} & \text{otherwise.} \end{cases}$$

An interpretation is that we draw "true" signals s' for the bidders from $\overline{\mathcal{S}}$ and agent i observes $s_i = \min\{\Delta\lfloor \Delta^{-1} s'_i \rfloor, \Delta^{-1}\}$, i.e., signals above Δ^{-1} are censored and otherwise they are rounded down to the nearest multiple of Δ, and w is the conditional expectation of $\overline{w}$ given the noisy observations s. Thus, the distribution of $\overline{w}$ is a mean-preserving spread of the distribution of w, so that H is a mean-preserving spread of the distribution of w as well.

Lemma 21. *If $s_i < \Delta^{-1}$ for all i, then $w(s)$ only depends on the sum of the signals $l = \Sigma s$ and*

$$w(s) = \frac{\exp(l)}{(1-\exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \overline{w}(x)\rho(x-l)\exp(-x)dx,$$

where $\rho(y)$ is the $N-1$-dimensional volume of the set $\{s \in [0,\Delta]^N | \Sigma s = y\}$.

Proof of Lemma 21. First observe that

$$f(s) = (1-\exp(-\Delta))^N \exp(-\Sigma s) = (1-\exp(-\Delta))^N \exp(-l).$$

Thus,

$$\begin{aligned} w(s) &= \frac{\exp(l)}{(1-\exp(-\Delta))^N} \int_{\left\{s' \in \mathbb{R}_+^N | \tau_i(s') = s_i \ \forall i\right\}} \overline{w}(\Sigma s')\exp(-\Sigma s')ds' \\ &= \frac{\exp(l)}{(1-\exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \int_{\left\{s' \in \mathbb{R}_+^N | \tau_i(s') = s_i \ \forall i, \Sigma s' = x\right\}} \overline{w}(\Sigma s')\exp(-\Sigma s')ds'dx \\ &= \frac{\exp(l)}{(1-\exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \overline{w}(x)\exp(-x) \int_{\left\{s' \in \mathbb{R}_+^N \middle| \tau_i(s'_i - s_i) = 0 \ \forall i, \Sigma s' = x\right\}} ds'dx \\ &= \frac{\exp(l)}{(1-\exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \overline{w}(x)\exp(-x) \int_{\left\{s' \in \mathbb{R}_+^N | \tau_i(s') = 0 \ \forall i, \Sigma s' = x-l\right\}} ds'dx, \end{aligned}$$

where the inner integral is just $\rho(x-l)$. □

We now abuse notation slightly by writing $w(l)$ for the value when $l = \Sigma s$, and let $\gamma(l) = w(l) - c$.

Lemma 22. *If $l > \Delta$, then $\gamma(l) \leq \exp(\Delta)\gamma(l-\Delta)$.*

Proof of Lemma 22. From Lemma 21, we know that

$$\gamma(l) = \frac{\exp(l)}{(1-\exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \overline{\gamma}(x)\exp(-x)\rho(x-l)dx$$

$$
\begin{aligned}
&= \frac{\exp(l)}{(1-\exp(-\Delta))^N} \int_{x=l-\Delta}^{l+(N-1)\Delta} \overline{\gamma}(x+\Delta)\exp(-x-\Delta)\rho(x-l+\Delta)dx \\
&\leq \frac{\exp(l-\Delta)}{(1-\exp(-\Delta))^N} \int_{x=l-\Delta}^{l+(N-1)\Delta} \overline{\gamma}(x)\exp(\Delta)\exp(-x)\rho(x-l+\Delta)dx \\
&= \exp(\Delta)\gamma(l-\Delta),
\end{aligned}
$$

where the inequality follows from Lemma 2. □

Lemma 23. *If the direct allocation $q_i(s)$ is Nash implemented by a participation secure mechanism, profit is at most*

$$
\sum_{s\in S} f(s) \sum_{i=1}^{N} q_i(s) \left[\gamma(\Sigma s) - \frac{1-F_i(s_i)}{f_i(s_i)}(\gamma(\Sigma s+\Delta)-\gamma(\Sigma s))\right]. \tag{40}
$$

Proof of Lemma 23. This follows from standard revenue equivalence arguments: If we write $U_i(s_i, s_i')$ for the utility of a signal s_i that reports s_i', with $U_i(s_i) = U_i(s_i, s_i)$, then

$$
U_i(s_i) \geq U_i(s_i, s_i') = U_i(s_i') + \sum_{s_{-i}\in S_{-i}} f_{-i}(s_{-i}) q_i(s_i', s_{-i}) \left(\gamma(s_i + \Sigma s_{-i}) - \gamma(s_i' + \Sigma s_{-i})\right).
$$

Thus, for $s_i \geq \Delta$,

$$
U_i(s_i) \geq U_i(0) + \sum_{k=0}^{s_i/\Delta-1} \sum_{s_{-i}\in S_{-i}} f_{-i}(s_{-i}) q_i(k\Delta, s_{-i}) \left(\gamma((k+1)\Delta + \Sigma s_{-i}) - \gamma(k\Delta + \Sigma s_{-i})\right).
$$

The expectation of $U_i(s_i)$ across s_i is therefore bounded below by

$$
\begin{aligned}
&\sum_{s\in S} f(s) \sum_{k=0}^{s_i/\Delta-1} q_i(k\Delta, s_{-i}) \left(\gamma((k+1)\Delta + \Sigma s_{-i}) - \gamma(k\Delta + \Sigma s_{-i})\right) \\
&= \sum_{s\in S} f(s) q_i(s) (\gamma(\Sigma s+\Delta) - \gamma(\Sigma s)) \frac{1-F_i(s_i)}{f_i(s_i)}.
\end{aligned}
$$

The formula then follows from subtracting the bound on bidder surplus from total surplus. □

Let Π denote the maximum of the profit bound (40) across all q. Let $\tilde{\Pi}$ denote the profit bound when we set $q_1(s) = 1$ and $q_j(s) = 0$ for all $j \neq 1$.

Lemma 24. $\Pi \leq \tilde{\Pi} + (1-(1-\exp(-\Delta^{-1}))^N)\overline{v}$.

Proof of Lemma 24. When signals are all less than Δ^{-1}, the bidder-independent virtual value is

$$
\gamma(l) - \frac{1}{\exp(\Delta)-1}\left(\gamma(l+\Delta) - \gamma(l)\right)
$$

$$\geq \gamma(l) - \frac{\exp(-\Delta)}{1-\exp(-\Delta)}(\gamma(l)\exp(\Delta) - \gamma(l)) = 0,$$

where the inequality follows from Lemma 22. Thus, the virtual value is maximized pointwise by allocating with probability one to, say, bidder 1. With probability $1-(1-\exp(-\Delta^{-1}))^N$, one of the signals is above Δ^{-1}, in which case $\overline{v}$ is an upper bound on the virtual value. □

Lemma 25. $\lim_{\Delta\to 0}\tilde{\Pi} \leq \overline{\Pi}$.

Proof of Lemma 25. Plugging in $q_1 = 1$, we find that

$$\begin{aligned}
\tilde{\Pi} &= \sum_{s_{-1}\in S_{-1}} f_{-1}(s_{-1}) \sum_{s_1\in S_1}\left[f_1(s_1)\gamma(\Sigma s) - \sum_{s_1'>s_1} f_1(s_1')(\gamma(\Sigma s+\Delta) - \gamma(\Sigma s))\right] \\
&= \sum_{s_{-1}\in S_{-1}} f_{-1}(s_{-1}) \sum_{s_1\in S_1}\left[f_1(s_1)\left[\gamma(\Sigma s) + \sum_{s_1'<s_1}(\gamma(s_1'+\Sigma s_{-1}) - \gamma(s_1'+\Sigma s_{-1}+\Delta))\right]\right] \\
&= \sum_{s_{-1}\in S_{-1}} f_{-1}(s_{-1})\gamma(\Sigma s_{-1}).
\end{aligned}$$

Using the definition of γ, this is

$$\begin{aligned}
\tilde{\Pi} &= \frac{1}{1-\exp(-\Delta)}\int_{y=0}^{\Delta}\int_{x=0}^{\infty}\overline{\gamma}(x+y)g_{N-1}(x)\exp(-y)dxdy \\
&= \frac{1}{1-\exp(-\Delta)}\int_{x=0}^{\infty}\overline{\gamma}(x)\int_{y=0}^{\min\{x,\Delta\}} g_{N-1}(x-y)\exp(-y)dydx \\
&\leq \frac{1}{1-\exp(-\Delta)}\left[\int_{x=\Delta}^{\infty}\overline{\gamma}(x)\int_{y=0}^{\Delta} g_{N-1}(x-y)\exp(-y)dydx + G_N(\Delta)\overline{v}\right].
\end{aligned}$$

Now, observe that

$$\begin{aligned}
\int_{y=0}^{\Delta} g_{N-1}(x-y)\exp(-y)dy &= \frac{x^{N-1}-(x-\Delta)^{N-1}}{(N-1)!}\exp(-x) \\
&\leq \frac{\Delta(N-1)x^{N-2}}{(N-1)!}\exp(-x) = \Delta g_{N-1}(x),
\end{aligned}$$

where we have used convexity of x^{N-1}. Thus,

$$\tilde{\Pi} \leq \frac{\Delta}{1-\exp(-\Delta)}\int_{x=0}^{\infty}\overline{\gamma}(x)g_{N-1}(x)dx + \frac{G_N(\Delta)\overline{v}}{1-\exp(-\Delta)}.$$

An application of L'Hôpital's rule shows that the last term converges to zero as $\Delta \to 0$ and $\Delta/(1-\exp(-\Delta)) \to 1$, this implies the lemma. □

Proof of Proposition 6. Combining Lemmas 23 and 24, we can pick Δ sufficiently small so that $\Pi \leq \tilde{\Pi} + \epsilon/2 \leq \overline{\Pi} + \epsilon$. This completes the proof of the proposition. □

Note that every step of the proof of Proposition 6 goes through in the must-sell case, where we replace $\overline{w}$ with $\widehat{w}$, and we skip the step in Lemma 24 of proving that the discrete virtual value is non-negative.

C Proofs for Section 6

Proof of Lemma 9. The left-tail assumption could equivalently be stated as: there exists some $\overline{\alpha} > 0$ and $\varphi > 1$ such that for all $0 \leq \alpha' < \alpha \leq \overline{\alpha}$

$$H^{-1}(\alpha) - \underline{v} \leq G_N^{-1}(\alpha)^{\varphi}$$

and if $\underline{v} > c$,

$$\frac{H^{-1}(\alpha) - c}{H^{-1}(\alpha') - c} \leq \exp(G_N^{-1}(\alpha) - G_N^{-1}(\alpha')).$$

The following Lemma 26 implies that if the above two conditions hold for N, they hold for all $N' > N$ as well. □

Lemma 26. *For any $N \geq 1$ and $N' > N$, there exists $\overline{\alpha} > 0$ such that $G_N^{-1}(\alpha) - G_N^{-1}(\alpha') \leq G_{N'}^{-1}(\alpha) - G_{N'}^{-1}(\alpha')$ for all $0 \leq \alpha' < \alpha \leq \overline{\alpha}$.*

Proof of Lemma 26. Clearly it suffices to prove the lemma for $N' = N + 1$. Let us extend the definition of G_N to any real number N:

$$G_N(x) = \int_{y=0}^{x} e^{-y} \frac{y^{N-1}}{\Gamma(N)}\, dy,$$

where

$$\Gamma(N) = \int_{y=0}^{\infty} e^{-y} y^{N-1}\, dy.$$

(We have $\Gamma(N) = (N-1)!$ when $N \geq 1$ is an integer.)

By definition, we have

$$\int_{x=0}^{G_N^{-1}(\alpha)} e^{-x} \frac{x^{N-1}}{\Gamma(N)}\, dx = \alpha.$$

Differentiating the above equation with respect to N gives:

$$\frac{\partial G_N^{-1}(\alpha)}{\partial N} \frac{e^{-G_N^{-1}(\alpha)} G_N^{-1}(\alpha)^{N-1}}{\Gamma(N)} + \int_{x=0}^{G_N^{-1}(\alpha)} e^{-x} \frac{\partial \left(\frac{x^{N-1}}{\Gamma(N)}\right)}{\partial N}\, dx = 0.$$

i.e.,

$$\begin{aligned}
\frac{\partial G_N^{-1}(\alpha)}{\partial N} =& \frac{\Gamma(N) e^{G_N^{-1}(\alpha)}}{G_N^{-1}(\alpha)^{N-1}} \left(- \int_{x=0}^{G_N^{-1}(\alpha)} e^{-x} \frac{\partial \left(\frac{x^{N-1}}{\Gamma(N)}\right)}{\partial N}\, dx \right) \\
=& \frac{e^{G_N^{-1}(\alpha)}}{\Gamma(N) G_N^{-1}(\alpha)^{N-1}} \int_{x=0}^{G_N^{-1}(\alpha)} e^{-x} [-x^{N-1} \log(x) \Gamma(N) + x^{N-1} \Gamma'(N)]\, dx \\
=& \frac{e^{G_N^{-1}(\alpha)}}{\Gamma(N)} f(G_N^{-1}(\alpha), N),
\end{aligned}$$

where

$$f(z,N) = \frac{1}{z^{N-1}} \int_{x=0}^{z} e^{-x}[-x^{N-1}\log(x)\Gamma(N) + x^{N-1}\Gamma'(N)]\,dx.$$

We compute:

$$\begin{aligned}\frac{\partial f(z,N)}{\partial z} =& \frac{1}{z^{2(N-1)}}\Bigg(z^{N-1}e^{-z}[-z^{N-1}\log(z)\Gamma(N) + z^{N-1}\Gamma'(N)] \\ &\qquad - (N-1)z^{N-2}\int_{x=0}^{z} e^{-x}[-x^{N-1}\log(x)\Gamma(N) + x^{N-1}\Gamma'(N)]\,dx\Bigg) \\ =& e^{-z}[-\log(z)\Gamma(N) + \Gamma'(N)] - (N-1)z^{-N}\int_{x=0}^{z} e^{-x}[-x^{N-1}\log(x)\Gamma(N) + x^{N-1}\Gamma'(N)]\,dx.\end{aligned}$$

For any $z \leq 1$, we have

$$\begin{aligned}\frac{\partial f(z,N)}{\partial z} \geq& e^{-z}[-\log(z)\Gamma(N) + \Gamma'(N)] - (N-1)z^{-N}\int_{x=0}^{z}[-x^{N-1}\log(x)\Gamma(N) + x^{N-1}\Gamma'(N)]\,dx \\ =& e^{-z}[-\log(z)\Gamma(N) + \Gamma'(N)] - (N-1)z^{-N}\left[\Gamma(N)\left(\frac{z^N}{N^2} - \frac{z^N\log z}{N}\right) + \Gamma'(N)\frac{z^N}{N}\right] \\ =& e^{-z}[-\log(z)\Gamma(N) + \Gamma'(N)] - \frac{N-1}{N}\left[\Gamma(N)\left(\frac{1}{N} - \log z\right) + \Gamma'(N)\right] \\ =& \left(e^{-z} - \frac{N-1}{N}\right)[-\log(z)\Gamma(N) + \Gamma'(N)] - \frac{N-1}{N^2}\Gamma(N).\end{aligned}$$

Since the last line goes to infinity as z goes to zero, for any fixed $N \geq 1$ we can choose $\overline{z} \in (0,1]$ such that $\partial f(z,\widehat{N})/\partial z \geq 0$ for all $z \in [0,\overline{z}]$ and $\widehat{N} \in [N, N+1]$. Let $\overline{\alpha} = G_{N+1}(\overline{z})$.

Suppose $0 \leq \alpha' < \alpha \leq \overline{\alpha}$. We have

$$[G_{N+1}^{-1}(\alpha) - G_{N+1}^{-1}(\alpha')] - [G_N^{-1}(\alpha) - G_N^{-1}(\alpha')] = \int_{\widehat{N}=N}^{N+1}\left(\frac{\partial G_{\widehat{N}}^{-1}(\alpha)}{\partial \widehat{N}} - \frac{\partial G_{\widehat{N}}^{-1}(\alpha')}{\partial \widehat{N}}\right)d\widehat{N}.$$

Since $d\left(e^z f(z,\widehat{N})/\Gamma(\widehat{N})\right)/dz \geq 0$ for all $z \in [0,\overline{z}]$ and $\widehat{N} \in [N, N+1]$, we have $\partial G_{\widehat{N}}^{-1}(\alpha)/\partial\widehat{N} - \partial G_{\widehat{N}}^{-1}(\alpha')/\partial\widehat{N} \geq 0$, which proves the lemma. □

Let us now define

$$\begin{aligned}G_N^C(x) &= G_N\left(\sqrt{N-1}x + N - 1\right); \\ g_N^C(x) &= \sqrt{N-1}\,g_N\left(\sqrt{N-1}x + N - 1\right).\end{aligned}$$

Note that this normalization differs slightly from that used in Section 6, as anticipated in Footnote 37. To prove Proposition 7, we first need a number of technical results.

Lemma 27. *As N goes to infinity, g_N^C and G_N^C converge pointwise to ϕ and Φ, respectively.*

Proof of Lemma 27. Note that

$$\begin{aligned} g^C_{N+1}(x) &= \sqrt{N} g_{N+1}(\sqrt{N}x + N) \\ &= \sqrt{N} \frac{(\sqrt{N}x + N)^N}{N!} \exp(-\sqrt{N}x - N). \end{aligned}$$

Stirling's Approximation says that

$$\lim_{N\to\infty} \frac{N!}{\sqrt{2\pi N}\left(\frac{N}{e}\right)^N} = 1.$$

Moreover, for all N, the ratio inside the limit is greater than 1.

Thus, when N is large, $g^C_{N+1}(x)$ is approximately

$$\frac{1}{\sqrt{2\pi}} \left(1 + \frac{x}{\sqrt{N}}\right)^N \exp(-\sqrt{N}x),$$

and hence

$$\log(g^C_{N+1}(x)) \approx \log(1/\sqrt{2\pi}) + N \log\left(1 + \frac{x}{\sqrt{N}}\right) - \sqrt{N}x.$$

Using the mean-value formulation of Taylor's Theorem centered around 0, for every y, there exists a $z \in [0, y]$ such that

$$\log(1 + y) = y - \frac{y^2}{2} + \frac{1}{(1+z)^3} y^3.$$

Plugging in $y = x/\sqrt{N}$, we conclude that

$$\begin{aligned} \log(g^C_{N+1}(x)) &\approx \log(1/\sqrt{2\pi}) + N\frac{x}{\sqrt{N}} - N\frac{1}{2}\left(\frac{x}{\sqrt{N}}\right)^2 + N\frac{1}{(1+z)^3}\left(\frac{x}{\sqrt{N}}\right)^3 - \sqrt{N}x \\ &= \log(1/\sqrt{2\pi}) - \frac{1}{2}x^2 + \frac{1}{(1+z)^3}\frac{x^3}{\sqrt{N}}, \end{aligned}$$

which converges to $\log(1/\sqrt{2\pi}) - \frac{1}{2}x^2$ as N goes to infinity, so $g^C_{N+1}(x)$ converges to $\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$. Pointwise convergence of G^C_N to Φ follows from Scheffé's lemma. □

Let us define

$$\tilde{g}(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) & \text{if } x < 0; \\ \frac{1}{\sqrt{2\pi}}(1+x)\exp(-x) & \text{otherwise.} \end{cases}$$

Lemma 28. *The function $\tilde{g}(x)|x|$ is integrable, and for all N and x, $|g^C_N(x)| \le \tilde{g}(x)$.*

Proof of Lemma 28. Note that

$$\int_{x=-\infty}^{\infty} \tilde{g}(x)|x|dx = \int_{x=-\infty}^{0} \phi(x)|x|dx + \frac{1}{\sqrt{2\pi}} \int_{x=0}^{\infty} (1+x)x\exp(-x)dx,$$

which is clearly finite, since the half-normal distribution has finite expectation.

Next, Stirling's Approximation implies that

$$g_{N+1}^C(x) \leq \frac{1}{\sqrt{2\pi}} \left(1 + \frac{x}{\sqrt{N}}\right)^N \exp(-\sqrt{N}x) \equiv \tilde{g}_N(x).$$

Now,

$$\frac{d}{dN} \log(\tilde{g}_N(x)) = \log\left(1 + \frac{x}{\sqrt{N}}\right) - \frac{1}{2}\frac{x}{\sqrt{N}+x} - \frac{x}{2\sqrt{N}},$$

which is clearly zero when $x = 0$, and

$$\begin{aligned}
\frac{d}{dx}\frac{d}{dN} \log(\tilde{g}_N(x)) &= \frac{1}{\sqrt{N}+x} - \frac{\sqrt{N}}{2(\sqrt{N}+x)^2} - \frac{1}{2\sqrt{N}} \\
&= \frac{2N + 2\sqrt{N}x}{2\sqrt{N}(\sqrt{N}+x)^2} - \frac{N}{2\sqrt{N}(\sqrt{N}+x)^2} - \frac{N + 2\sqrt{N}x + x^2}{2\sqrt{N}(\sqrt{N}+x)^2} \\
&= \frac{-x^2}{2\sqrt{N}(\sqrt{N}+x)^2},
\end{aligned}$$

which is non-positive and strictly negative when $x \neq 0$. As a result, $\tilde{g}_N(x)$ is increasing in N when $x < 0$ and decreasing in N when $x > 0$. Since it converges to $\phi(x)$ in the limit as N goes to infinity, we conclude that for $x < 0$, $g_{N+1}^C(x) \leq \tilde{g}_N(x) \leq \phi(x) = \tilde{g}(x)$, and for $x > 0$, $g_{N+1}^C(x) \leq \tilde{g}_N(x) \leq \tilde{g}_1(x) = \tilde{g}(x)$ as desired. □

Lemma 29. *As N goes to infinity, $\widehat{\gamma}_N^C$ converges almost surely to $\widehat{\gamma}_\infty^C(x) = H^{-1}(\Phi(x))$ and $\widehat{\Gamma}_N^C$ converges pointwise to*

$$\widehat{\Gamma}_\infty^C(x) = \int_{y=-\infty}^{x} \widehat{\gamma}_\infty^C(y)\phi(y)dy.$$

The latter convergence is uniform on any bounded interval.

Proof of Lemma 29. Note that $\widehat{\gamma}_N^C(x) = H^{-1}(G_N^C(x)) - c$. By Lemma 27, $G_N^C(x)$ converges to $\Phi(x)$ pointwise. Thus, if H^{-1} is continuous at $\Phi(x)$, then as N goes to infinity, we must have $\widehat{\gamma}_N^C(x) \to H^{-1}(\Phi(x)) - c = \widehat{\gamma}_\infty^C(x)$. Since H^{-1} is monotonic, the set of discontinuities has Lebesgue measure zero, so that the pointwise convergence is almost everywhere.

Pointwise convergence of $\widehat{\Gamma}_N^C$ follows from almost sure convergence of $\widehat{\gamma}_N^C$, combined with the fact that $\widehat{\gamma}_N^C$ is uniformly bounded by $|\overline{v}|$, so that we can apply the dominated convergence theorem. Moreover, $\widehat{\Gamma}_N^C(x)$ is uniformly Lipschitz continuous across N and x. As a result, the family $\{\widehat{\Gamma}_N^C(\cdot)\}_{N=2}^\infty$ is uniformly bounded and uniformly equicontinuous. The conclusion about uniform convergence is then a consequence of the Arzela-Ascoli theorem. □

Recall that x^* is the largest solution to $\widehat{\Gamma}^C_\infty(x^*) = 0$ (which may be $-\infty$). Also, let us define x_N so that $\overline{\Gamma}^C_N$ has a graded interval $[-\sqrt{N-1}, x_N]$. (If there is no graded interval with left end point $-\sqrt{N-1}$, then we let $x_N = -\sqrt{N-1}$.)

Lemma 30. *As N goes to infinity, x_N converges to x^*.*

Proof of Lemma 30. By a change of variables $y = (G^C_N)^{-1}(\Phi(x))$, we conclude that

$$\widehat{\Gamma}^C_\infty(x^*) = \int_{x=-\infty}^{x^*} \widehat{\gamma}^C_\infty(x)\phi(x)dx = \int_{x=-\sqrt{N-1}}^{(G^C_N)^{-1}(\Phi(x^*))} \widehat{\gamma}^C_N(x) g^C_N(x)dx = \widehat{\Gamma}^C_N\left((G^C_N)^{-1}(\Phi(x^*))\right).$$

This integral must be zero by the definition of x^*, so that $x_N \geq (G^C_N)^{-1}(\Phi(x^*))$. Since the latter converges to x^* as $N \to \infty$, we conclude $\liminf_{N\to\infty} x_N \geq x^*$.

Next, recall that x_{N+1} solves the equation

$$\begin{aligned}
\widehat{\Gamma}^C_{N+1}(x_{N+1}) &= \widehat{\gamma}^C_{N+1}(x_{N+1}) \int_{x=-\sqrt{N}}^{x_{N+1}} \exp(\sqrt{N}(x - x_{N+1})) g^C_{N+1}(x)dx \\
&= \widehat{\gamma}^C_{N+1}(x_{N+1}) \exp(-\sqrt{N}x_{N+1} - N) \int_{x=-\sqrt{N}}^{x_{N+1}} \exp(\sqrt{N}x + N) g^C_{N+1}(x)dx \\
&= \widehat{\gamma}^C_{N+1}(x_{N+1}) \exp(-\sqrt{N}x_{N+1} - N) \int_{x=-\sqrt{N}}^{x_{N+1}} \sqrt{N}\frac{(\sqrt{N}x + N)^N}{N!}dx \\
&\leq \overline{v} \exp(-\sqrt{N}x_{N+1} - N)\frac{(\sqrt{N}x_{N+1} + N)^{N+1}}{(N+1)!} \\
&= \overline{v} g^C_{N+2}\left(\sqrt{\frac{N}{N+1}}x_{N+1} - \frac{1}{\sqrt{N+1}}\right)\frac{1}{\sqrt{N+1}} \\
&\leq \overline{v}\tilde{g}\left(\sqrt{\frac{N}{N+1}}x_{N+1} - \frac{1}{\sqrt{N+1}}\right)\frac{1}{\sqrt{N+1}},
\end{aligned}$$

where we have used Lemma 28. The last line converges to zero pointwise, so $\widehat{\Gamma}^C_N(x_N)$ must converge to zero as well.

Now, if $z = \limsup_{N\to\infty} x_N > x^*$, then since $\widehat{\Gamma}^C_\infty(z) > \widehat{\Gamma}^C_\infty(x^*) = 0$, we would contradict our earlier finding that $\widehat{\Gamma}^C_N(x_N) \to 0$. Thus, $\limsup_{N\to\infty} x_N \leq x^*$, so x_N must converge to x^* as N goes to ∞. □

Lemma 31. *For every $\epsilon > 0$, there exists $\widehat{N}$ such that for all $N > \widehat{N}$, there exists an $x \in [x^* + \epsilon, x^* + 2\epsilon]$ at which $\overline{\gamma}^C_N$ is not graded.*

Proof of Lemma 31. Suppose not. Then there exist infinitely many N such that for every $x \in [x^* + \epsilon, x^* + 2\epsilon]$, $\overline{\gamma}^C_{N+1}(x) = \exp(\sqrt{N}(x - \tilde{x}))\widehat{\gamma}^C_{N+1}(\tilde{x})$ for some $\tilde{x} \geq x^* + 2\epsilon$. Thus, for all $x \leq x^* + \epsilon$, we conclude that

$$\overline{\gamma}^C_{N+1}(x) \leq \overline{\gamma}^C_{N+1}(x^* + \epsilon) \leq \exp(-\sqrt{N}\epsilon)\overline{v}$$

which converges to zero as N goes to infinity. This implies that $\liminf_{N\to\infty} \overline{\Gamma}^C_{N+1}(x^*+\epsilon) = 0$. But $\overline{\Gamma}^C_{N+1}(x^*+\epsilon)$ must be weakly larger than $\widehat{\Gamma}^C_{N+1}(x^*+\epsilon)$, so

$$0 = \lim \inf_{N\to\infty} \overline{\Gamma}^C_{N+1}(x^*+\epsilon) \geq \lim \inf_{N\to\infty} \widehat{\Gamma}^C_{N+1}(x^*+\epsilon) = \widehat{\Gamma}^C_{\infty}(x^*+\epsilon) > 0,$$

a contradiction. □

Lemma 32. *As N goes to infinity, $\overline{\gamma}^C_N$ converges almost surely to*

$$\overline{\gamma}^C_\infty(x) = \begin{cases} 0 & \text{if } x < x^*; \\ \widehat{\gamma}^C_\infty(x) & \text{if } x \geq x^*. \end{cases}$$

Proof of Lemma 32. Let $x < x^*$. Since $x_N \to x^*$ by Lemma 30, for N sufficiently large, $x_N > (x^*+x)/2$. Since $\overline{\gamma}^C_N(x)$ is graded on $(-\infty, x_N]$, it is graded at x, and

$$\begin{aligned} \overline{\gamma}^C_N(x) &= \exp(\sqrt{N-1}(x-x_N))\widehat{\gamma}^C_N(x_N) \\ &\leq \exp(\sqrt{N-1}(x-x^*)/2)\overline{v}. \end{aligned}$$

The last line clearly converges to zero pointwise. Since $\overline{\gamma}^C_N(x) \geq 0$ for all N, we conclude that $\overline{\gamma}^C_N(x) \to 0$.

Now consider $x > x^*$ at which $\widehat{\gamma}^C_\infty$ is continuous. Take ϵ so that $x > x^* + 2\epsilon$ and so that $\widehat{\gamma}^C_\infty$ is continuous at $x^*+\epsilon$. Lemma 31 says that there is a $\widehat{N}$ such that for all $N > \widehat{N}$, there exists a point in $[x^*+\epsilon, x^*+2\epsilon]$ at which the gains function is not graded. Moreover, since $\widehat{\gamma}^C_N(x^*+\epsilon)$ converges to $\widehat{\gamma}^C_\infty(x^*+\epsilon)$, we can pick $\widehat{N}$ large enough and find a constant $\underline{\gamma} > 0$ such that for $N > \widehat{N}$, $\widehat{\gamma}^C_N(x^*+\epsilon) \geq \underline{\gamma}$.

Now, suppose that $\overline{\gamma}^C_N$ is graded at x, with x in a graded interval $[a,b]$. Then $a \geq x^*+\epsilon$, and hence $\widehat{\gamma}^C_N(a) \geq \widehat{\gamma}^C_N(x^*+\epsilon) \geq \underline{\gamma}$. Recall that on $[a,b]$,

$$\overline{\gamma}^C_N(x) = \widehat{\gamma}^C_N(a)\exp(\sqrt{N-1}(x-a)).$$

Since $\widehat{\gamma}^C_N$ is bounded above by $\overline{v}$, it must be that $\widehat{\gamma}^C_N(a)\exp(\sqrt{N-1}(b-a)) \leq \overline{v}$, so

$$\begin{aligned} b-a &\leq \frac{1}{\sqrt{N-1}} \log\left(\frac{\overline{v}}{\widehat{\gamma}^C_N(a)}\right) \\ &\leq \frac{1}{\sqrt{N-1}} \log\left(\frac{\overline{v}}{\underline{\gamma}}\right) = \epsilon_N. \end{aligned}$$

Thus,

$$\widehat{\gamma}^C_N(x-\epsilon_N) \leq \overline{\gamma}^C_N(x) \leq \widehat{\gamma}^C_N(x+\epsilon_N).$$

This was true if $\overline{\gamma}^C_N(x)$ is graded at x, but clearly the inequality is also true if it is not graded at x, in which case $\overline{\gamma}^C_N(x) = \widehat{\gamma}^C_N(x)$. Now, $\widehat{\gamma}^C_N(x) = \widehat{\gamma}^C_\infty(\Phi^{-1}(G^C_N(x)))$, so

$$\widehat{\gamma}^C_\infty(\Phi^{-1}(G^C_N(x-\epsilon_N))) \leq \overline{\gamma}^C_N(x) \leq \widehat{\gamma}^C_\infty(\Phi^{-1}(G^C_N(x+\epsilon_N))).$$

As $N \to \infty$, the left and right hand sides converge to $\widehat{\gamma}^C_\infty(x)$ from the left and right, respectively. Since $\widehat{\gamma}^C_\infty$ is continuous at x, we conclude that $\overline{\gamma}^C_N(x) \to \widehat{\gamma}^C_\infty(x)$. The lemma follows from the fact that the monotonic function $\widehat{\gamma}^C_\infty$ is continuous almost everywhere. □

Proof of Proposition 7. We argue that

$$Z_{N+1} = \sqrt{N} \int_{x=0}^{\infty} \overline{\gamma}_{N+1}(x)(g_{N+1}(x) - g_N(x))dx$$

converges to a positive constant as N goes to infinity. Since this is $\sqrt{N}$ times the difference between ex ante gains from trade and profit, this proves the result.

To that end, observe that

$$Z_{N+1} = \sqrt{N} \int_{x=0}^{N/2} \overline{\gamma}_{N+1}(x)(g_{N+1}(x) - g_N(x))dx + \int_{x=-\sqrt{N}/2}^{\infty} \overline{\gamma}^C_{N+1}(x) g^C_{N+1}(x) \frac{Nx}{\sqrt{N}x + N} dx.$$

We claim that the first integral converges to zero as $N \to \infty$. Note that $g_{N+1}(x) \leq g_N(x)$ if and only if $x \leq N$. Therefore,

$$\begin{aligned}
\left| \sqrt{N} \int_{x=0}^{N/2} \overline{\gamma}_{N+1}(x)(g_{N+1}(x) - g_N(x))dx \right| &\leq (\overline{v} + c)\sqrt{N} \int_{x=0}^{N/2} (g_N(x) - g_{N+1}(x))dx \\
&= (\overline{v} + c)\sqrt{N}(G_N(N/2) - G_{N+1}(N/2)) \\
&= (\overline{v} + c)\sqrt{N} g_{N+1}(N/2) \\
&= (\overline{v} + c)\sqrt{N} \frac{(N/2)^N \exp(-N/2)}{N!} \\
&\approx (\overline{v} + c)\sqrt{N} \frac{(N/2)^N \exp(-N/2)}{\sqrt{2\pi N}(N/e)^N} \\
&= (\overline{v} + c) \frac{1}{\sqrt{2\pi}} \exp(-N(\log(2) - 1/2)),
\end{aligned}$$

where we have again used Stirling's Approximation between the third-to-last and second-to-last lines. The last line converges to zero as N goes to infinity.

Now consider the second integral in the formula for Z_{N+1}. By Lemma 28, the integrand is bounded above in absolute value by the integrable function $\overline{v}\tilde{g}(x)|x|$. Moreover, from Lemmas 27 and 32, we know that the integrand converges pointwise to $\overline{\gamma}^C_\infty(x)\phi(x)x$. The dominated convergence theorem then implies that as N goes to infinity, Z_N converges to

$$\int_{x=-\infty}^{\infty} \overline{\gamma}^C_\infty(x)\phi(x)x dx,$$

which is strictly positive because $\overline{\gamma}^C_\infty$ is strictly increasing.

The proof goes through for the must-sell guarantee, if we replace $\overline{\gamma}^C_N$ with $\widehat{\gamma}^C_N$. □

To prove Proposition 9, we need a few more intermediate results. Let $\overline{G}_N(x) = G_N(Nx)$ be the cumulative distribution for the mean of N independent standard exponential random variables. Define $\overline{F}_N(x) = \exp(N(1 - x + \log(x)))$. Clearly, $\overline{F}_N(x)$ is a cumulative distribution for $x \in [0, 1]$, $\overline{F}_N(0) = 0$ and $\overline{F}_N(1) = 1$. Finally, define the function $D_N(\alpha)$:

$$D_N(\alpha) = \begin{cases} \frac{1}{\overline{F}_N^{-1}(\alpha)} & \text{if } \alpha \in [0, 0.4]; \\ 1.1 & \text{if } \alpha \in (0.4, 1]. \end{cases}$$

The choices of 0.4 and 1.1 in $D_N(\alpha)$ are arbitrary: any numbers work that are less than 1/2 and more than 1, respectively.

Lemma 33. *When $\widehat{N}$ is sufficiently large, $\overline{\mu}_N(G_N^{-1}(\alpha)) \leq D_{\widehat{N}}(\alpha)$ for all $N \geq \widehat{N}$ and $\alpha \in [0,1]$.*

Proof of Lemma 33. We first apply the theory of large deviations to the exponential distribution. Let $\Lambda(t)$ be the logarithmic moment generating function for the exponential distribution:

$$\Lambda(t) = \log\left(\int_{x=0}^{\infty} \exp(xt - x)\,dx\right) = \begin{cases} \infty & \text{if } t \geq 1; \\ -\log(1-t) & \text{if } t < 1. \end{cases}$$

Let $\Lambda^*(x)$ be the Legendre transform of $\Lambda(t)$:

$$\Lambda^*(x) = \sup_{t\in\mathbb{R}}\{xt - \Lambda(t)\} = \begin{cases} \infty & x \leq 0, \\ x - 1 - \log x & x > 0. \end{cases}$$

Cramér's theorem (or the Chernoff bound; see Theorem 1.3.12 in Stroock, 2011) then states that for any N,

$$\overline{G}_N(x) \leq \exp(-N\Lambda^*(x)) = \overline{F}_N(x)$$

for every $x \in [0,1]$; or equivalently, $\overline{F}_N^{-1}(\alpha) \leq \overline{G}_N^{-1}(\alpha)$ for every $\alpha \in [0, \overline{G}_N(1)]$.

By the law of large numbers, when $\widehat{N}$ is sufficiently large, we have $\overline{G}_N(1) \geq 0.4$ and $1/\overline{G}_N^{-1}(0.4) \leq 1.1$ and for all $N \geq \widehat{N}$. The claim of the lemma then follows from two cases:

If $\alpha \in [0, 0.4]$, then we have

$$\overline{\mu}_N(G_N^{-1}(\alpha)) \leq \frac{N}{G_N^{-1}(\alpha)} = \frac{1}{\overline{G}_N^{-1}(\alpha)} \leq \frac{1}{\overline{F}_N^{-1}(\alpha)} \leq \frac{1}{\overline{F}_{\widehat{N}}^{-1}(\alpha)} = D_{\widehat{N}}(\alpha),$$

where we have used the bound $\overline{\mu}_N(x) \leq N/x$ (equation (21)), and the facts that $\overline{G}_N(1) \geq 0.4$ when $N \geq \widehat{N}$ (so $\overline{F}_N^{-1}(\alpha) \leq \overline{G}_N^{-1}(\alpha)$ for $\alpha \leq 0.4 \leq \overline{G}_N(1)$) and that $\overline{F}_N(x) \leq \overline{F}_{\widehat{N}}(x)$ for all $N \geq \widehat{N}$ and $x \in [0,1]$ (so $\overline{F}_{\widehat{N}}^{-1}(\alpha) \leq \overline{F}_N^{-1}(\alpha)$ for all α).

If $\alpha \in (0.4, 1]$, then

$$\overline{\mu}_N(G_N^{-1}(\alpha)) \leq \frac{1}{\overline{G}_N^{-1}(\alpha)} \leq \frac{1}{\overline{G}_N^{-1}(0.4)} \leq 1.1 = D_{\widehat{N}}(\alpha),$$

since $\overline{G}_N^{-1}(\alpha)$ is increasing in α, and $1/\overline{G}_N^{-1}(0.4) \leq 1.1$ when $N \geq \widehat{N}$. □

Lemma 34. *When N is sufficiently large,*

$$\int_{\alpha=0}^{1} D_N(\alpha)\,dH^{-1}(\alpha) < \infty.$$

Proof of Lemma 34. Since $G_N(x) = 1 - \sum_{k=1}^{N} g_k(x)$, we have:

$$\begin{aligned}\overline{G}_N(x) &= 1 - \sum_{k=1}^{N} \exp(-Nx)\frac{(Nx)^{k-1}}{(k-1)!} \\ &= 1 - \exp(-Nx)\left(\exp(Nx) - \sum_{k=N}^{\infty}\frac{(Nx)^k}{k!}\right) \geq \exp(-Nx)\frac{(Nx)^N}{N!}.\end{aligned}$$

Clearly, there exists an $\overline{x} \in (0,1)$ such that

$$\overline{F}_{N+1}(x) = \exp((N+1)(1-x))x^{N+1} \leq \exp(-Nx)\frac{(Nx)^N}{N!} \leq \overline{G}_N(x)$$

for all $x \in [0, \overline{x}]$. We therefore have $D_{N+1}(\alpha) = 1/\overline{F}_{N+1}^{-1}(\alpha) \leq 1/\overline{G}_N^{-1}(\alpha)$ for all $\alpha \in [0, \overline{\alpha}]$, where $\overline{\alpha} = \min\{\overline{F}_{N+1}(\overline{x}), 0.4\}$. As a result,

$$\int_{\alpha=0}^{1} D_{N+1}(\alpha)\, dH^{-1}(\alpha) \leq \int_{\alpha=0}^{\overline{\alpha}} \frac{1}{\overline{G}_N^{-1}(\alpha)}\, dH^{-1}(\alpha) + \int_{\alpha=\overline{\alpha}}^{1} \max\left(\frac{1}{\overline{F}_{N+1}^{-1}(\overline{\alpha})}, 1.1\right) dH^{-1}(\alpha) < \infty$$

whenever we have

$$\int_{\alpha=0}^{1} \frac{1}{\overline{G}_N^{-1}(\alpha)}\, dH^{-1}(\alpha) = \int_{x=0}^{\infty} \frac{N}{x}\, d\widehat{w}_N(x) < \infty.$$

Finiteness of the last integral follows from part one of the left-tail assumption. □

Lemma 35. *Suppose* $\lim_{N\to\infty} y_N \in (-\infty, \infty)$. *Then* $\lim_{N\to\infty} \overline{\mu}_{N+1}(\sqrt{N}y_N + N) = 1$.

Proof of Lemma 35. We first argue that for almost every y, $\overline{\mu}_{N+1}(\sqrt{N}y + N)$ tends to 1 as $N \to \infty$. For this we recall x^* and x_N from Lemmas 30–32.

Consider first $y < x^*$. By Lemma 30, for N sufficiently large, the gains function is graded at y, and hence

$$\overline{\mu}_{N+1}(\sqrt{N}y + N) = C(0, \sqrt{N}x_{N+1} + N) = \frac{N+1}{\sqrt{N}x_{N+1} + N}.$$

Since we have already shown that $x_N \to x^*$ (Lemma 30), we conclude that $\overline{\mu}_{N+1}(\sqrt{N}y + N)$ goes to 1.

Now consider $y > x^*$ at which $\widehat{\gamma}_\infty^C$ is continuous. If the gains function is not graded at y, then $\overline{\mu}_{N+1}(\sqrt{N}y + N) = N/(\sqrt{N}y + N)$. If the gains function is graded at y, then the length of the graded interval $[a, b] \ni y$ in the central limit units is less than $\epsilon_N = \overline{v}/(\underline{\gamma}\sqrt{N})$ for some $\underline{\gamma} > 0$ independent of N (see Lemma 32). Since $\overline{\mu}$ is decreasing (Lemma 3), we have

$$\frac{N}{\sqrt{N}(y + \epsilon_N) + N} \leq \overline{\mu}_{N+1}(\sqrt{N}y + N) \leq \frac{N}{\sqrt{N}(y - \epsilon_N) + N},$$

since $\lim_{z\nearrow a} \overline{\mu}_{N+1}(\sqrt{N}z + N) = N/(\sqrt{N}a + N)$ and $\lim_{z\searrow b} \overline{\mu}_{N+1}(\sqrt{N}z + N) = N/(\sqrt{N}b + N)$. As a result, $\overline{\mu}_{N+1}(\sqrt{N}y + N)$ is squeezed to 1 as N goes to infinity.

We conclude that $\overline{\mu}_{N+1}(\sqrt{N}y+N)$ goes to 1 for $y > x^*$ at which $\widehat{\gamma}^C_\infty$ is continuous. Since $\widehat{\gamma}^C_\infty(y)$ is a monotone function of y, it is continuous at almost every y, so the convergence $\overline{\mu}_N \to 1$ is almost everywhere.

Finally, suppose $\lim_{N\to\infty} y_N = y \in (-\infty, \infty)$. Choose y' and y'' such that $y \in (y', y'')$ and such that

$$\lim_{N\to\infty} \overline{\mu}_{N+1}(\sqrt{N}y' + N) = 1 = \lim_{N\to\infty} \overline{\mu}_{N+1}(\sqrt{N}y'' + N).$$

When N is sufficiently large, we have $y_N \in (y', y'')$, so

$$\overline{\mu}_{N+1}(\sqrt{N}y'' + N) \le \overline{\mu}_{N+1}(\sqrt{N}y_N + N) \le \overline{\mu}_{N+1}(\sqrt{N}y' + N).$$

Taking the limit as $N \to \infty$, we conclude $\lim_{N\to\infty} \overline{\mu}_{N+1}(\sqrt{N}y_N + N) = 1$. □

Proof of Proposition 9. We first prove that

$$\lim_{N\to\infty} \overline{\lambda}_N(v; H) \to v - c \tag{41}$$

for every $v \in [\underline{v}, \overline{v}]$.

Replacing $\overline{\mu}_N$ by 1 in equation (18), the definition of $\overline{\lambda}_N(v; H)$, we have

$$\begin{aligned}\overline{\Pi}_N(H) + \int_{y=0}^{\infty} G_N(y)\, d\widehat{w}_N(y) - \int_{\nu=v}^{\overline{v}} d\nu &= \overline{\Pi}_N(H) + \left(\overline{v} - \int_{y=0}^{\infty} g_N(y)\widehat{w}_N(y)\, dy\right) - (\overline{v} - v) \\ &= \overline{\Pi}_N(H) - \int_{v'=\underline{v}}^{\overline{v}} v'\, dH(v') + v.\end{aligned}$$

Since by Proposition 7 $\lim_{N\to\infty} \overline{\Pi}_N(H) \to \int_{v'=\underline{v}}^{\overline{v}} v'\, dH(v') - c$, to prove (41), it suffices to prove that

$$\lim_{N\to\infty} \int_{y=0}^{\infty} |1 - \overline{\mu}_N(y)|\, d\widehat{w}_N(y) = 0.$$

Changing variables, we can rewrite the above equation as:

$$\lim_{N\to\infty} \int_{\alpha=0}^{1} |1 - \overline{\mu}_N(G_N^{-1}(\alpha))|\, dH^{-1}(\alpha) = 0. \tag{42}$$

We note that Stieltjes integration with respect to $dH^{-1}(\alpha)$ is equivalent to a Lebesgue integration with respect to the finite measure ω on $[0, 1]$ satisfying $\omega([s, t)) = H^{-1}(t) - H^{-1}(s)$, $0 \le s \le t \le 1$, and $\omega(\{1\}) = 0$. Part one of the left-tail assumption implies that

$$\omega(\{0\}) = \lim_{\alpha\to 0} \omega([0, \alpha)) = \lim_{\alpha\to 0} H^{-1}(\alpha) - H^{-1}(0) \le \lim_{\alpha\to 0} G_N^{-1}(\alpha)^{\varphi} = 0$$

for some $\varphi > 1$. Therefore, $\omega(\{0, 1\}) = 0$.

The central limit theorem implies that $\lim_{N\to\infty}(G_N^{-1}(\alpha)-(N-1))/\sqrt{N-1} = \Phi^{-1}(\alpha)$ for every $\alpha \in (0, 1)$. Therefore, Lemma 35 implies $\lim_{N\to\infty} \overline{\mu}_N(G_N^{-1}(\alpha)) = 1$ for every $\alpha \in (0, 1)$. Moreover, Lemmas 33 and 34 imply that there exists a $\widehat{N}$ such that for all $N \ge \widehat{N}$, the

integrand $|1-\overline{\mu}_N(G_N^{-1}(\alpha))|$ in (42) is dominated by $1+D_{\widehat{N}}(\alpha)$ which is integrable with respect to ω. Therefore, equation (42) follows from the dominated convergence theorem, from which equation (41) follows.

Finally, using the definition of $\overline{\lambda}_N(v;H)$, we have

$$\overline{\lambda}_N(v;H) \leq \overline{\Pi}_N(H)+\int_{y=0}^{\infty} \overline{\mu}_N(y)(1+G_N(y))\,d\widehat{w}_N(y) \leq (\overline{v}-c)+2\int_{\alpha=0}^{1} D_{\widehat{N}}(\alpha)\,dH^{-1}(\alpha) < \infty,$$

for all $v \in [\underline{v},\overline{v}]$ and $N \geq \widehat{N}$, where the last two inequalities follow from Lemmas 33 and 34, respectively. Thus

$$\lim_{N\to\infty} \int_V \overline{\lambda}_N(v;H)\,dH'(v) = \int_V v\,dH'(v) - c$$

follows the dominated convergence theorem using (41).

The proof for the must-sell $\widehat{\lambda}_N(v;H)$ is identical, after replacing $\overline{\mu}_N(x)$ with $\widehat{\mu}_N(x) = (N-1)/x$ and $\overline{\Pi}_N(H)$ with $\widehat{\Pi}_N(H)$. □

Lemma 36. *Suppose the condition on H in Lemma 10 holds. For any $\epsilon > 0$, there exists an $\widehat{N}$ such that for all $N > \widehat{N}$, we have*

$$\widehat{\gamma}_N(x) \leq \widehat{\gamma}_N(y)\exp(x-y).$$

for all $x \geq y$ such that $\widehat{\gamma}_N(y) \geq \epsilon$.

Proof of Lemma 36. The condition on H implies that the support of H has no gap on $[\underline{v},\overline{v}]$, so H^{-1} is continuous on $[0,1]$. We can partition $[0,1]$ into a countable collection of intervals $\{[\alpha_i,\beta_i] : i \in I\}$ such that $\alpha_i < \beta_i$, and either H^{-1} is strictly increasing on $[\alpha_i,\beta_i]$, or H^{-1} is constant on $[\alpha_i,\beta_i]$ (i.e., H has a mass point at v, where $v = H^{-1}(p)$ for all $p \in [\alpha_i,\beta_i]$). If H^{-1} is strictly increasing on $[\alpha_i,\beta_i]$, then

$$H^{-1}(q) - H^{-1}(p) \leq \frac{q-p}{C}. \tag{43}$$

for any $p,q \in (\alpha_i,\beta_i)$ such that $p \leq q$, since in this case we have $H(H^{-1}(q)) = q$ and $H(H^{-1}(p)) = p$. By continuity of H^{-1} we can extend (43) to any $p,q \in [\alpha_i,\beta_i]$ such that $p \leq q$.

If H^{-1} is constant on $[\alpha_i,\beta_i]$, then clearly (43) also holds for any $p,q \in [\alpha_i,\beta_i]$ such that $p \leq q$. Since $\{[\alpha_i,\beta_i] : i \in I\}$ is a partition of $[0,1]$, we conclude that (43) holds for any $p,q \in [0,1]$ such that $p < q$.

With the substitution $q = G_N^C(x)$ and $p = G_N^C(y)$, with $x > y$, equation (43) becomes

$$\widehat{\gamma}_N^C(x) - \widehat{\gamma}_N^C(y) \leq \frac{G_N^C(x) - G_N^C(y)}{C}.$$

Thus,

$$\frac{\widehat{\gamma}_N^C(x)}{\widehat{\gamma}_N^C(y)} \leq 1 + \frac{1}{\widehat{\gamma}_N^C(y)}\frac{G_N^C(x) - G_N^C(y)}{C}.$$

The log-1 Lipschitz condition that we want to prove is equivalent to

$$\frac{\widehat{\gamma}_N^C(x)}{\widehat{\gamma}_N^C(y)} \leq \exp(G_N^{-1}(G_N^C(x)) - G_N^{-1}(G_N^C(y))).$$

Thus, it is sufficient to show that for large N,

$$1 + \frac{1}{\widehat{\gamma}_N^C(y)} \frac{G_N^C(x) - G_N^C(y)}{C} \leq \exp(G_N^{-1}(G_N^C(x)) - G_N^{-1}(G_N^C(y))).$$

Both sides are equal to one when $x = y$, and the derivatives of the left- and right-hand sides with respect to x are, respectively

$$\frac{g_N^C(x)}{\widehat{\gamma}_N^C(y)C}, \tag{44}$$

and

$$\begin{aligned} &\frac{g_N^C(x)}{g_N(G_N^{-1}(G_N^C(x)))} \exp(G_N^{-1}(G_N^C(x)) - G_N^{-1}(G_N^C(y))) \\ &= \sqrt{N-1} \exp(G_N^{-1}(G_N^C(x)) - G_N^{-1}(G_N^C(y))) \geq \sqrt{N-1}. \end{aligned} \tag{45}$$

We now show that (44) is always less than (45). Note that g_N attains its maximum when $g_N = g_{N-1}$, i.e., when $x = N-1$, at a value of $\frac{(N-1)^{N-1}}{(N-1)!} \exp(-(N-1))$. Multiplied by $\sqrt{N-1}$, this upper bound converges to $\phi(0)$. Hence, when N is sufficiently large, $g_N^C(x) \leq 2\phi(0)$ for all x. Since $\widehat{\gamma}_N^C(z) > 0$, then there is an N large enough such that

$$\frac{g_N^C(x)}{\widehat{\gamma}_N^C(y)C} \leq \frac{2\phi(0)}{\epsilon C} \leq \sqrt{N-1}$$

which proves the lemma. □

Proof of Lemma 10. If $\underline{v} > c$, then we can take $\epsilon = \underline{v} - c$ in the statement of Lemma 36, in which case the statement of the Lemma follows immediately.

If $\underline{v} < c$, then $\widehat{\gamma}_N^C(-\sqrt{N-1}) < 0$, so that $\widehat{\Gamma}_N^C(x)$ is non-positive for x close to $-\sqrt{N-1}$. Hence, there must be a graded interval at the bottom of the form $[-\sqrt{N-1}, x_N]$. By Lemma 30, x_N converges to x^*. Moreover, by Lemma 32, $\overline{\gamma}_N^C$ converges almost surely to $\overline{\gamma}_\infty^C$. Thus, there exists an $\widehat{N}$ such that for all $N > \widehat{N}$, $\widehat{\gamma}_N^C(x_N) \geq \epsilon$. If we take $\epsilon = \widehat{\gamma}_\infty^C(x^*)/2$ in Lemma 36, then there exists a $\widehat{N}' \geq \widehat{N}$ so that for all $N > \widehat{N}'$, the log-1 Lipschitz condition is satisfied for all $x \geq x_N$. This implies that there is exactly one graded interval, and the conclusion of the Lemma follows. □

Proof of Proposition 10. We first derive the allocation. When $\underline{v} > c$, we have $x^* = -\infty$ and the gains function $\overline{\gamma}$ is not graded when N is sufficiently large. In this case $\overline{Q}_N^C(x)$ is always exactly 1.

When $\underline{v} < c$, $x^* \in (-\infty, \infty)$, and the gains function $\overline{\gamma}$ is single crossing (Section 4.4) when N is sufficiently large. Then $\overline{Q}_N^C(x) = \min((x\sqrt{N} + N)/(x_N\sqrt{N} + N), 1)$. Since x_N converges to x^* as defined by equation (29), $\overline{Q}_N^C(x)$ converges to 1 as $N \to \infty$.

We now derive the transfer. From Lemma 10, we know that there is at most one graded interval of the form $[-\sqrt{N}, x_N]$, where $x_N = -\sqrt{N}$ if $\underline{v} > c$ and $x_N > -\sqrt{N}$ if $\underline{v} < c$.

Recall that

$$\overline{T}_N(x) = \frac{1}{g_N(x)} \int_{y=0}^{x} \overline{\Xi}_N(y) g_N(y)\, dy,$$

$$\overline{\Xi}_N(x) = \overline{\mu}_N(x)\widehat{w}_N(x) - \overline{\lambda}_N(\widehat{w}_N(x)) - c\overline{Q}_N(x),$$

$$\begin{aligned}\overline{\lambda}_N(\widehat{w}_N(x)) &= \int_{y=0}^{\infty} \overline{\gamma}_N(y) g_{N-1}(y) dy + \int_{y=0}^{\infty} \overline{\mu}_N(y) G_N(y) d\widehat{w}_N(y) - \int_{y=x}^{\infty} \overline{\mu}_N(y) d\widehat{w}_N(y) \\ &= \int_{y=0}^{\infty} \overline{\gamma}_N(y) g_{N-1}(y) dy + \int_{y=0}^{\infty} \overline{\mu}_N(y) G_N(y) d\widehat{w}_N(y) + \overline{\mu}_N(x)\widehat{w}_N(x) + \int_{y=x}^{\infty} \widehat{w}_N(y) d\widehat{\mu}_N(y).\end{aligned}$$

Furthermore,

$$\begin{aligned}\int_{y=0}^{\infty} \overline{\mu}_N(y) G_N(y) d\widehat{w}_N(y) &= \int_{y=0}^{\infty} \overline{\mu}_N(y) G_N(y) d\widehat{\gamma}_N(y) \\ &= -\int_{y=0}^{\infty} \widehat{\gamma}_N(y) d(\overline{\mu}_N(y) G_N(y)) \\ &= -\int_{y=0}^{\infty} \widehat{\gamma}_N(y) G_N(y) d\overline{\mu}_N(y) - \int_{y=0}^{\infty} \widehat{\gamma}_N(y) \overline{\mu}(y) g_N(y) dy \\ &= -\int_{y=0}^{\infty} \widehat{\gamma}_N(y) G_N(y) d\overline{\mu}_N(y) - \int_{y=0}^{\infty} \overline{\gamma}_N(y) g_{N-1}(y) dy,\end{aligned}$$

where the last inequality comes from equation (32). Thus,

$$\overline{\lambda}_N(\widehat{w}_N(x)) = -\int_{y=0}^{\infty} \widehat{\gamma}_N(y) G_N(y) d\overline{\mu}_N(y) + \overline{\mu}_N(x)\widehat{w}_N(x) + \int_{y=x}^{\infty} \widehat{w}_N(y) d\overline{\mu}_N(y),$$

and

$$\begin{aligned}\overline{\Xi}_N(x) &= \int_{y=0}^{x} \widehat{\gamma}_N(y) G_N(y) d\overline{\mu}_N(y) + \int_{y=x}^{\infty} (\widehat{\gamma}_N(y) G_N(y) - \widehat{w}_N(y)) d\overline{\mu}_N(y) - c\overline{Q}_N(x) \\ &= \int_{y=0}^{x} \widehat{\gamma}_N(y) G_N(y) d\overline{\mu}_N(y) - \int_{y=x}^{\infty} \widehat{\gamma}_N(y)(1 - G_N(y)) d\overline{\mu}_N(y) - c(\overline{Q}_N(x) - \overline{\mu}_N(x))\end{aligned}$$

Let us now switch to central limit units.

$$\begin{aligned}\Xi_N^C(x) &= \overline{\Xi}_N(\sqrt{N-1}x + N - 1) \\ &= \int_{y=-\sqrt{N}}^{x} \widehat{\gamma}_N^C(y) G_N^C(y) d\overline{\mu}_N^C(y) - \int_{y=x}^{\infty} \widehat{\gamma}_N^C(y)(1 - G_N^C(y)) d\overline{\mu}_N^C(y) - c(\overline{Q}_N^C(x) - \overline{\mu}_N^C(x)).\end{aligned}$$

By Lemmas 27 and 29, $\widehat{\gamma}_N^C(y) \to \widehat{\gamma}_\infty^C(y) = H^{-1}(\Phi(y)) - c$ and $G_N^C(y) \to \Phi(y)$ as $N \to \infty$.

Moreover, we have

$$\sqrt{N-1}d\overline{\mu}_N^C(y) = \begin{cases} 0 & \text{if } y < x_N; \\ (N-1)\left(\frac{N-1}{x_N\sqrt{N-1}+N-1} - \frac{N}{x_N\sqrt{N-1}+N-1}\right) \to -1 & \text{if } y = x_N; \\ -(N-1)\frac{N-1}{(y\sqrt{N-1}+N-1)^2}dy \to -dy & \text{if } y > x_N, \end{cases}$$

where the mass point on x_N is derived by comparing $\overline{\mu}_N^C$ to the left and right of x_N, and

$$\sqrt{N-1}(\overline{Q}_N^C(x) - \overline{\mu}_N^C(x)) = \begin{cases} \sqrt{N-1}\left(\frac{x\sqrt{N-1}+N-1}{x_N\sqrt{N-1}+N-1} - \frac{N}{x_N\sqrt{N-1}+N-1}\right) & \text{if } x < x_N; \\ \sqrt{N-1}\left(1 - \frac{N-1}{x\sqrt{N-1}+N-1}\right) & \text{if } x > x_N, \end{cases}$$

which converges to x in both cases.

Define $F(x) = \lim_{N\to\infty} \sqrt{N-1}\,\overline{\Xi}_N^C(x)$. We have

$$F(x) = \begin{cases} -cx + \widehat{\gamma}_\infty^C(x^*)(1-\Phi(x^*)) + \int_{y=x^*}^\infty \widehat{\gamma}_\infty^C(y)(1-\Phi(y))\,dy & x < x^* \\ -cx - \widehat{\gamma}_\infty^C(x^*)\Phi(x^*) - \int_{y=x^*}^x \widehat{\gamma}_\infty^C(y)\Phi(y)\,dy + \int_{y=x}^\infty \widehat{\gamma}_\infty^C(y)(1-\Phi(y))\,dy & x > x^* \end{cases}.$$

Therefore,

$$\lim_{N\to\infty} \overline{T}_N^C(x) = \frac{1}{\phi(x)} \int_{y=0}^x F(y)\phi(y)\,dy.$$

□

D Derivation of Aggregate Transfer for Uniform Distribution

Suppose the prior H is the standard uniform distribution, so that $\widehat{w}(x) = G_N(x)$, and that $c = 0$.

D.1 Must-sell Case

In the must-sell case, $\widehat{\Xi}$ and $\widehat{T}$ are independent of c, so $c = 0$ is without loss. We have:

$$\begin{aligned}
\widehat{\lambda}(G_N(x)) &= \int_{y=0}^{\infty} G_N(y) g_{N-1}(y)\, dy + \int_{y=0}^{\infty} \frac{N-1}{y} G_N(y) g_N(y)\, dy - \int_{y=x}^{\infty} \frac{N-1}{y} g_N(y)\, dy \\
&= 2\int_{y=0}^{\infty} G_N(y) g_{N-1}(y)\, dy - (1 - G_{N-1}(x)) \\
&= 2\widehat{\Pi} - (1 - G_{N-1}(x)), \\
\widehat{\Xi}(x) &= \frac{N-1}{x} G_N(x) - G_{N-1}(x) + 1 - 2\widehat{\Pi}.
\end{aligned}$$

Next,

$$\begin{aligned}
\int_{y=0}^{x} \widehat{\Xi}(y) g_N(y)\, dy &= \int_{y=0}^{x} \left(\frac{N-1}{y} G_N(y) - G_{N-1}(y) + 1 - 2\widehat{\Pi} \right) g_N(y)\, dy \\
&= 2\int_{y=0}^{x} G_N(y) g_{N-1}(y)\, dy - G_N(x) G_{N-1}(x) + (1 - 2\widehat{\Pi}) G_N(x) \\
&= G_{N-1}(x)^2 - 2\int_{y=0}^{x} g_N(y) g_{N-1}(y)\, dy - G_N(x) G_{N-1}(x) + (1 - 2\widehat{\Pi}) G_N(x) \\
&= G_{N-1}(x) g_N(x) - 2\int_{y=0}^{x} g_N(y) g_{N-1}(y)\, dy + (1 - 2\widehat{\Pi}) G_N(x) \\
&= G_{N-1}(x) g_N(x) - \frac{(2N-3)!}{2^{2N-3}(N-1)!(N-2)!} G_{2N-2}(2x) + (1 - 2\widehat{\Pi}) G_N(x) \\
&= G_{N-1}(x) g_N(x) + \frac{(2N-3)!}{2^{2N-3}(N-1)!(N-2)!} (G_N(x) - G_{2N-2}(2x))
\end{aligned}$$

where the second line follows from integration by parts, the third and fourth lines use $G_N = G_{N-1} - g_N$, the fifth line is a direct computation using the formula for g_N in (14), and the last line follows from

$$\widehat{\Pi} = \int_{y=0}^{\infty} G_N(y) g_{N-1}(y)\, dy = \frac{1}{2} - \int_{y=0}^{\infty} g_N(y) g_{N-1}(y)\, dy = \frac{1}{2}\left(1 - \frac{(2N-3)!}{2^{2N-3}(N-1)!(N-2)!}\right).$$

Therefore, when $x > 0$,

$$\widehat{T}(x) = G_{N-1}(x) + \frac{\binom{2N-3}{N-1}}{2^{2N-3}} \frac{G_N(x) - G_{2N-2}(2x)}{g_N(x)}.$$

In the central limit normalization (using the convention in Section C), we define

$$\widehat{T}^C(x) = \widehat{T}(N-1+\sqrt{N-1}x).$$

Lemma 27 shows that $G_N(N-1+\sqrt{N-1}x) \to \Phi(x)$ and $g_N(N-1+\sqrt{N-1}x)\sqrt{N-1} \to \phi(x)$ as $N \to \infty$, where Φ and ϕ are, respectively, the cumulative distribution and density of a standard Normal; this also implies that $G_{2N-2}(2(N-1+\sqrt{N-1}x)) \to \Phi(x\sqrt{2})$. Finally, using Stirling's approximation, it is easy to check that $\frac{\binom{2N-3}{N-1}}{2^{2N-3}}\sqrt{N-1} \to \frac{1}{\sqrt{\pi}}$ as $N \to \infty$. Therefore,

$$\lim_{N\to\infty} \widehat{T}^C(x) = \Phi(x) + \frac{\Phi(x) - \Phi(x\sqrt{2})}{\sqrt{\pi}\,\phi(x)}$$

for a fixed x.

D.2 Can-keep Case

We have shown that the uniform distribution is single-crossing in Section 4.4. Let $[0, x^*]$ denote the graded interval. The cutoff x^* satisfies (cf. (28))

$$\frac{G_N(x^*)}{2} = g_{N+1}(x^*). \tag{46}$$

This equation implies that $G_{N+1}(x^*) = G_N(x^*) - g_{N+1}(x^*) = g_{N+1}(x^*) = G_N(x^*)/2$.

Define the constants

$$\begin{aligned}
C =& \int_{x=0}^{\infty} \overline{\gamma}(x) g_{N-1}(x)\,dx + \int_{x=0}^{\infty} \overline{\mu}(x) G_N(x) g_N(x)\,dx \\
=& \underbrace{\int_{x=0}^{x^*} \exp(x-x^*) G_N(x^*) g_{N-1}(x)\,dx + \int_{x=0}^{x^*} \frac{N}{x^*} G_N(x) g_N(x)\,dx}_{C_1} \\
&+ \underbrace{\int_{x=x^*}^{\infty} G_N(x) g_{N-1}(x)\,dx + \int_{x=x^*}^{\infty} \frac{N-1}{x} G_N(x) g_N(x)\,dx}_{C_2}
\end{aligned}$$

We can simplify the constants as follows:

$$\begin{aligned}
C_1 =& 2\int_{x=0}^{x^*} \exp(x-x^*) G_N(x^*) g_{N-1}(x)\,dx \\
=& 2 G_N(x^*) g_N(x^*) \\
C_2 =& 2\int_{x=x^*}^{\infty} G_N(x) g_{N-1}(x)\,dx \\
=& 1 - G_{N-1}(x^*)^2 - 2\int_{x=x^*}^{\infty} g_N(x) g_{N-1}(x)\,dx \\
=& 1 - G_{N-1}(x^*)^2 - \frac{\binom{2N-3}{N-1}}{2^{2N-3}}(1 - G_{2N-2}(2x^*))
\end{aligned}$$

$$C = 2G_N(x^*)g_N(x^*) + 1 - G_{N-1}(x^*)^2 - \frac{\binom{2N-3}{N-1}}{2^{2N-3}}(1 - G_{2N-2}(2x^*)).$$

Then

$$\begin{aligned}
\overline{\lambda}(G_N(x)) &= C - \int_{y=x}^{\infty} \overline{\mu}(y) g_N(y)\, dy \\
&= \begin{cases} C - \int_{y=x}^{x^*} \frac{N}{x^*} g_N(y)\, dy - \int_{y=x^*}^{\infty} \frac{N-1}{y} g_N(y)\, dy & x \le x^* \\ C - \int_{y=x}^{\infty} \frac{N-1}{y} g_N(y)\, dy & x > x^* \end{cases} \\
&= \begin{cases} C - (G_N(x^*) - G_N(x))\frac{N}{x^*} - (1 - G_{N-1}(x^*)) & x \le x^* \\ C - (1 - G_{N-1}(x)) & x > x^* \end{cases}
\end{aligned}$$

and

$$\overline{\Xi}(x) = \begin{cases} G_N(x)\frac{N}{x^*} - C + (G_N(x^*) - G_N(x))\frac{N}{x^*} + (1 - G_{N-1}(x^*)) & x \le x^* \\ \qquad\qquad = -C + G_N(x^*)\frac{N}{x^*} + 1 - G_{N-1}(x^*) & \\ G_N(x)\frac{N-1}{x} - C + 1 - G_{N-1}(x) & x > x^* \end{cases}$$

For $x \le x^*$, we have:

$$\begin{aligned}
\int_{y=0}^{x} \overline{\Xi}(y) g_N(y)\, dy &= \int_{y=0}^{x} \left(-C + G_N(x^*)\frac{N}{x^*} + 1 - G_{N-1}(x^*) \right) g_N(y)\, dy \\
&= \left(-C + G_N(x^*)\frac{N}{x^*} + 1 - G_{N-1}(x^*) \right) G_N(x).
\end{aligned}$$

For $x > x^*$, we have:

$$\begin{aligned}
\int_{y=0}^{x} \overline{\Xi}(y) g_N(y)\, dy = &\left(-C + G_N(x^*)\frac{N}{x^*} + 1 - G_{N-1}(x^*) \right) G_N(x^*) \\
&+ \underbrace{\int_{x^*}^{x} \left(G_N(y)\frac{N-1}{y} - C + 1 - G_{N-1}(y) \right) g_N(y)\, dy}_{X}.
\end{aligned}$$

Simplifying the second term, we get:

$$\begin{aligned}
X =& (1 - C)(G_N(x) - G_N(x^*)) \\
&+ 2\int_{y=x^*}^{x} G_N(y) g_{N-1}(y) dy - (G_N(x)G_{N-1}(x) - G_N(x^*)G_{N-1}(x^*)) \\
=& (1 - C)(G_N(x) - G_N(x^*)) \\
&- 2\int_{y=x^*}^{x} g_N(y) g_{N-1}(y) dy + g_N(x)G_{N-1}(x) - g_N(x^*)G_{N-1}(x^*) \\
=& (1 - C)(G_N(x) - G_N(x^*))
\end{aligned}$$

$$- \frac{\binom{2N-3}{N-1}}{2^{2N-3}}(G_{2N-2}(2x) - G_{2N-2}(2x^*)) + g_N(x)G_{N-1}(x) - g_N(x^*)G_{N-1}(x^*).$$

Therefore, for $x \leq x^*$, we have:

$$\overline{T}(x) = \left(-C + G_N(x^*)\frac{N}{x^*} + 1 - G_{N-1}(x^*)\right)\frac{G_N(x)}{g_N(x)}.$$

For $x > x^*$ we have:

$$\begin{aligned}&\overline{T}(x)\\ &= \left[G_N(x^*)^2\frac{N}{x^*} - G_{N-1}(x^*)^2 + (1-C)G_N(x) - \frac{\binom{2N-3}{N-1}}{2^{2N-3}}(G_{2N-2}(2x) - G_{2N-2}(2x^*))\right]\frac{1}{g_N(x)} + G_{N-1}(x).\end{aligned}$$

Finally, we take the limit as $N \to \infty$ for the central limit normalization:

$$\overline{T}^C(x) = \overline{T}(N - 1 + \sqrt{N-1}x).$$

Since $G_N(x^*)/2 = G_{N+1}(x^*)$ by the discussion following equation (46), we must have $(x^* - (N-1))/\sqrt{N-1} \to -\infty$, $G_N(x^*) \to 0$, and $g_N(x^*) \to 0$ as $N \to \infty$. Moreover, by equation (46), $NG_N(x^*)/x^* = 2Ng_{N+1}(x^*)/x^* = 2g_N(x^*) \to 0$ as $N \to \infty$. Substituting these into the expressions of C and $\overline{T}$ and simplify as in the must-sell case, we get

$$\lim_{N\to\infty} \overline{T}^C(x) = \Phi(x) + \frac{\Phi(x) - \Phi(x\sqrt{2})}{\sqrt{\pi}\,\phi(x)}.$$