An Informationally-Robust Market Model of Perfect Competition*

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Abstract

A large number of buyers with single unit demand have a common value for a good being sold. Buyers decide whether or not they wish to purchase the good; the goods are rationed among those who wish to purchase; and the market price is a function of the number of buyers who wish to purchase. We characterize pricing rules for which, as the number of buyers grows large, the expected market price converges to the expected value, regardless of the buyers' information and equilibrium strategies: these are pricing rules that have vanishing price impact and are asymptotically inelastic. Interpreting the pricing rule as a market supply function, we also prove that as long as the pricing rule has vanishing price impact, then in the large market, welfare is at least at the level when the buyers have no information about the value. We extend our results to the case where there is also an idiosyncratic component to the value and where buyers have multi-unit demands.

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1 Introduction

A nearly ubiquitous assumption in models of competitive markets is that traders perfectly observe prices before deciding how to trade. As a result, trading behavior can incorporate all of the information contained in prices, as a signal about the knowledge of the individual traders. This is one of the key ideas, along with price-taking behavior, that is expressed in rational expectations equilibrium (REE). But the assumption that trade is conditioned on prices is obviously not correct in practice, for at least two reasons. For one thing, traders may choose not to fully respond to prices because it is of cognitively burdensome. This issue is being investigated in the developing literature on rational inattention (Sims, 2003).[1] Another basic issue is that real consumption choices are made in a rich dynamic process, and along the way, consumers make decisions that partially commit themselves to trade. And while these decisions can depend on some information about future prices, they can not be conditioned directly on prices that are ultimately realized.

To motivate this fact, consider two examples: In the first, a couple chooses a restaurant to patronize for a date night. They look at descriptions of restaurants, which have coarse ratings for cost, represented by one to three dollar signs. They even glance through menus and observe a subset of prices. They make a reservation. When the evening arrives, the wait staff brings the menu for their final consideration. They order food and wine and enjoy the rest of their evening. As analysts, we may ask: when were consumption choices made? Did they condition on prices? There is no simple answer: the final order was made after seeing the full menu, but the choice of restaurant was made with partial information about prices. Clearly, once they were seated, the couple could not change the venue without paying a very high cost (or possibly forgoing dinner altogether). Thus, even if the menu turned out to be pricier than expected, they would still purchase dinner, perhaps with some marginal adjustments, e.g., whether to order chicken or steak for the entree.

For the second example, a family decides to go on a road trip. Halfway through the trip they have to fill the car up with gas. At the planning stage, they do not know exactly where they will stop, and even if they did, they would not be able to perfectly predict gas prices days in advance. Suppose that mid-trip the tank is low and gas prices are higher than expected. Do they abandon the car and walk home? Surely, in this situation, it is reasonable to model the decision to buy gas as being made at the beginning of the trip and *before* the price is known. And yet, the family knew the *expected* price, and took this into account in deciding to go on the trip.

[1] A prominent example of consumers' rational inattention to prices is the widely documented "left-digit bias" (see, e.g., List et al., 2023, and references therein).

Of course, we could enrich the standard model to account for all of the dynamic structure and imperfect information involved in market behavior. But this comes at a high cost in terms of analytical tractability; like much of the literature, we want to maintain a static metaphor for complex trading behavior, but with one key difference: we want to allow for the possibility that consumers may not know all of the factors that determine prices when they decide what to consume.

A distinct issue with the standard model of competitive markets is the assumption that agents assume they have exactly zero price impact, which cannot possibly be true in practice. A large literature, which we review below, has taken up the challenge of describing competitive markets as a limit of finite markets, where agents have positive (but perhaps vanishingly small) price impact. With such an approach, one is forced to model how prices are formed as a *result* of traders' decisions. The approach taken in this literature is to model traders' as placing *limit orders*, whereby they effectively condition their trades on the eventual market price.

In this paper, we fully embrace the idea that consumers may not know all of the relevant factors that determine prices when deciding how to trade. Moreover, we analyze behavior in large but finite markets, where each individual trader has positive price impact. The key question of interest is what happens to welfare as the market grows large.

To be more specific, in our model, a large number goods are available to be sold to a large number of buyers. The buyers have a common value for the good, and private and differential information. Each buyer must decide, based on their own private information, whether or not to purchase the good. Thus, in our model, agents place *market orders*, in contrast to the limit orders studied in the prior literature. A market price is then determined as a function of the aggregate order (i.e., the number of buyers who wish to purchase). The pricing function can either be thought of as a rule chosen by a monopolistic seller, or as a reduced form of a competitive supply side. Finally, buyers who place orders receive the good at the market price, with the good being rationed if the number of orders exceeds the number of units available.

The model could be interpreted as a metaphor for any kind of market, such as meals at restaurants or tankfulls of gas, where we simplify our representation of the consumption decision as either to buy or not buy. The model can also apply to commodities like bushels of wheat, or to shares of common stock. For these applications, it would be more natural to suppose that buyers can purchase more than one unit. As we will explain later, all of our results immediately extend to the case where buyers can consume multiple units, and utility is linear in consumption, up to some maximum amount, as long as each buyer's maximum demand is small relative to the rest of the market. But for most of the paper, we focus on the single-unit demand, which is notationally simpler environment in which to exposit the key ideas.

Our results characterize the set of possible welfare outcomes that could obtain under such *market order mechanisms*, as we range across all models of the buyers' information and Bayes Nash equilibria. We are especially interested in the limit when the number of buyers goes to infinity, as an approximation of welfare in large markets. A standard intuition is that in this limit, buyers will compete away all their rents, and expected revenue-per-unit will converge to the expected value. A key question is whether or not the limit welfare outcome is "competitive" in this sense, that the goods sell out and expected buyer surplus converges to zero.

In general, the welfare outcome in the large-market limit will depend on the sequence of pricing functions, and as we illustrate with examples, the limit outcome need not be competitive. There are two key frictions: First, if the price were to jump up at a particular number of orders, then the economy could end up in an equilibrium where the number of orders and the market price are at the lower side of the jump, and the expected value is strictly above the market price. However, no more buyers want to purchase, as that would push the price up above the expected value. In such a situation, buyers would still obtain rents, even when there are infinitely many of them, and sales may be inefficiently low.

Second, even if the price varies smoothly with the amount demanded, then there is still scope for the aggregate order to be correlated with the value in such a manner as to depress prices and revenue. The key issue is that when the aggregate order can vary with the value, then average expected value among buyers who place orders can be very different from the average expected value among buyers who do not place orders. This could in turn support non-competitive outcomes where the market price is strictly below the values of buyers who place orders but strictly above the values of those who do not place orders.

Our first main result formalizes the role played by these anticompetitive phenomena. We provide a lower bound on expected revenue that depends on three terms: (i) the pricing rule's *price impact*, which is the maximum amount by which a single order can change the price; (ii) the *window of price discovery*, which is the set of aggregate orders (as a fractions of the population) at which the market price takes on intermediate values; and (iii) the number of buyers. The lower is the price impact, the smaller is the window of price discovery, and the larger is the number of buyers, the closer is the price per unit sold to the expected value.

We say that a sequence of pricing rules (indexed by the number of buyers) has *vanishing price impact* if the price impact goes to zero. The sequence is *asymptotically inelastic* if the window of price discovery converges to a point, meaning that the range of aggregate orders for which the price is intermediate grows strictly slower than the number of buyers. An immediate corollary of our main result is that if a sequence of pricing rules has vanishing price impact and is asymptotically inelastic, then in the limit, the expected price-per-unit-sold converges to the expected value, and buyers compete away all of their rents. In addition, as long as the window of price discovery is centered on a point that is sufficiently high, then the good will almost surely sell out. It is in this sense that market order mechanisms are "competitive" in the limit. Finally, this convergent result holds *regardless of the sequence of information structures and equilibria*. As we describe below, these headline results extend to the case where there is an idiosyncratic component to buyers' values and to settings where buyers have multi-unit demand.

There are two interpretations of our theorem, corresponding to the aforementioned interpretations of the pricing rule: In one interpretation, there is a monopolistic seller who can produce multiple units at zero cost. The seller chooses the pricing function and commits to sell via a market order mechanism. In that setting, our results imply that in the limit of a large market, the seller can extract all of the surplus, no matter the buyers' information and equilibrium strategies. This generalizes a finding of Brooks and Du (2021) when there was a single unit for sale to the case where there are many units for sale, buyers have multi unit demand and idiosyncratic components of values, and the number of available units can grow with the size of the market. We also show that full surplus extraction can be achieved within the relatively simple class of market order mechanisms.

In the second interpretation, the buyers interact in a decentralized market. After making their purchase decisions, a pool of (at least two) sellers compete via Bertrand competition to attract buyers and fill orders. The pricing rule is the market supply curve. In this setting, our theorem shows that if the supply curve is asymptotically inelastic and has vanishing price impact, then the market outcome is efficient, and buyers compete away all rents. Moreover, the efficient and competitive outcome is obtained regardless of the buyers' information or which equilibrium is played.

The assumption of vanishing price impact seems relatively innocuous, but the asymptotic inelasticity, by contrast, seems quite demanding and particular. And as our examples show, if the market supply curve is elastic, then the equilibrium outcome may be socially efficient, and buyers need not compete away their rents. It is important to note that in our model, if the supply curve is not perfectly inelastic, then depending on the buyers' information, it may be *infeasible* to implement the ex post efficient outcome, simply because the buyers may not collectively know the value (which must be known in order to determine the efficient level of production). A natural benchmark for welfare is the total surplus that would be realized if the buyers had no information at all, except for knowing the prior distribution of the value. A priori, it seems plausible that equilibrium welfare could be even lower than this benchmark, due to the aforementioned frictions. Our second main result shows that even though the market outcome may be ex post or even interim inefficient, social surplus cannot fall below the no information benchmark. Thus, regardless of the form of private information and the equilibrium, private information is always welfare enhancing relative to no information.

Our modeling of behavior in decentralized markets may be contrasted with other approaches taken in the literature. In particular, in REE, it is presumed that each trader observes the market price before deciding whether or not to trade, and moreover, that traders understand the equilibrium relationship between prices and fundamentals. In our model, the buyers may not know the price at the time they decide whether or not to trade. Since our positive results hold across all information structures and equilibria, they do cover those instances where buyers know what will be the equilibrium aggregate order, and hence the market price. But in our negative results, it is certainly the case that buyers might wish to change their actions if they knew the eventual market price.

The simultaneous determination of prices and trades in REE has long been a source of discomfort among economists, and a substantial literature has attempted to reconcile this conceptual quandary by explicitly modeling large but finite markets (Wilson, 1977; Milgrom, 1979; Pesendorfer and Swinkels, 1997; Kremer, 2002; Bali and Jackson, 2002; Reny and Perry, 2006). This literature on "microfounding" REE has relied on auction-like mechanisms, such as first-price auctions or double auctions. In such mechanisms, a trader's action is related to a price at which they are willing to trade, and in that sense they function more like "limit orders." Whether or not behavior in these mechanisms converges to REE depends on the assumed sequence of information structures and also on the particular sequence of equilibria being played. It is by now well understood that the limit outcome of these mechanisms need not be an REE or even competitive (Engelbrecht-Wiggans, Milgrom, and Weber, 1983; Bergemann, Brooks, and Morris, 2017; Barelli, Govindan, and Wilson, 2023). Unlike much of this literature, our objective is not to justify or microfound REE. Rather, we take the market order mechanism seriously as a metaphor for market interactions. For the case of asymptotically inelastic pricing rules with vanishing price impact, we obtain competitive outcomes in the limit, although the limit behavior need not be a REE.

Pushing beyond our headline results, we also demonstrate that market order mechanisms can achieve competitive outcomes in three extensions of our baseline model. In the first extension, we consider buyers who have both common and private components to their values; each buyer knows their private component and has partial and differential information about the common value component, as in our baseline model. Under the hypotheses of vanishing price impact and asymptotic inelasticity, we show that as the number of buyers grows large, the expected price in market order mechanisms converges to the expected value of the marginal buyer, i.e., the market clearing price at which demand equals the supply. Moreover, under the assumption of vanishing price impact, limiting welfare in the decentralized market must be at least the optimal welfare when all buyers have no information beyond their private value components.

The second extension allows for uncertainty in the number of buyers and the number of units that are available, in addition to uncertainty about the value. We show market order mechanisms also eliminate any winner's curse that might arise through correlation between the number of potential buyers and the value, such as that described in Lauermann and Wolinsky (2017, 2022). Thus, the competitive outcome is still obtained even when the value and the numbers of buyers and units are both uncertain and correlated.

Finally, as mentioned above, we extend our results to the case where each buyer may demand more than one unit, and may have heterogeneous demands, but where the number of units demanded is uniformly bounded, independent of the size of the market. This extension allows us to capture within our model markets for commodities like wheat or petroleum, as well as sales of common stock. Indeed, one application of market order mechanisms would be for the sale of shares of equity in an initial public offering, where now the market order is placed for a number of units of the good. While it significantly expands the range of applications, this extension turns out to be mathematically trivial: multi-unit demand has the effect of scaling up the perceived price impact, but price impact still vanishes in the limit.

Methodologically, the present paper is an application of the framework for informationally-robust mechanism design developed in Brooks and Du (2024). In particular, the proof of our main result proceeds by computing a lower bound on expected revenue, where the lower bound is an expected (over states) lowest (over action profiles) *strategic virtual objective*. This tool was introduced in Brooks and Du (2024), and represents a kind of dual counterpart of the virtual value that is familiar from auction theory.[2] In general, the strategic virtual objective is defined, for each given action profile and payoff-relevant state, to be the designer's objective plus the changes in agents' utilities from "local" deviations away from the action that corresponds to opting out. In market order mechanisms, there are only two actions, buy and not buy. This leads to an especially simple and tractable form for the strategic virtual objective.

The rest of this paper is organized as follows. Section 2 describes our model. Section 3 presents our results on revenue maximization by a monopolist seller. Section 4 contains results on decentralized markets. Section 5 presents results with both common and private components in value. Section 6 extends to multi-unit demands. Section 7 is a discussion and

[2]More precisely, the strategic virtual objective is the dual counterpart to the *informational virtual objective*, which is a generalization of the virtual value that was introduced in Myerson (1981).

conclusion. An appendix contains omitted proofs as well as additional results when values, the number of buyers, and the number of goods may all be uncertain and correlated.

2 Market Order Mechanisms

There are K units of a good available for trade. There are also N buyers, each of whom demands a single unit of the good.

The buyers have a pure common value for the good denoted $v \in V = [\underline{v}, \overline{v}] \subseteq \mathbb{R}_+$. The distribution of common values is denoted $\mu \in \Delta(V)$.

The buyers' private information about the common value is described by an *information structure* $I = (S, \sigma)$, where S_i is a finite set of signals (or types) for buyer i, $S = \prod_i S_i$, and $\sigma \in \Delta(V \times S)$ is the joint distribution of the values and signals. We let $\mathcal{I}(\mu)$ be the set of information structures for which $\operatorname{marg}_V \sigma = \mu$.

Throughout the paper, we focus on a particular class of *market order mechanisms* by which trade occurs: Each buyer takes an action $a_i \in \{0, 1\}$. Given an action profile $a \in A = \{0, 1\}^N$, let $n = \sum_i a_i \equiv \Sigma a$ be the aggregate demand. The probability that buyer i receives a unit is $a_i r(n)$, where

$$r(n) \equiv \min\{K/n, 1\}.$$

In addition, buyers who receive a unit pay a price $p(n)$, where $p : \{0, 1, \ldots, N\} \to [0, \mathcal{C}]$ is a pricing rule, where $\mathcal{C}$ is a constant that is larger than $\overline{v}$. For our results we hold $\underline{v}, \overline{v}$ and $\mathcal{C}$ fixed as we increase N.

For mechanisms of this form, one can interpret $a_i = 1$ as a market order to buy one unit of the good, at whatever is the prevailing market price. The market price, in turn, is a function of the aggregate order. All buyers who place a market order will be assigned a unit of the good if there is excess supply, or assigned a unit randomly if there is excess demand. Moreover, note that if a buyer does not place an order, $a_i = 0$, then that buyer does not receive a unit and does not pay anything. Hence, market order mechanisms satisfy the notion of *participation security* of Brooks and Du (2021, 2024). We may contrast market order mechanisms with first- or second-price auctions, where a bid should be interpreted as a limit order to buy a unit at a given price (cf. Jovanovic and Menkveld, 2022).

Given the information structure I, a *strategy* for buyer i is a mapping $b_i : S_i \to \Delta(\{0, 1\})$, that assigns to each signal s_i a likelihood of placing an order $b_i(1|s_i)$. Buyer i's expected utility given a strategy profile b and the pricing rule p is

$$U_i(p, I, b) \equiv \int_{v,s,a} a_i r(\Sigma a)(v - p(\Sigma a)) \left(\prod_i b_i(a_i|s_i) \right) \sigma(dv, ds).$$

A *(Bayes-Nash) equilibrium* for the game (p, I) is a strategy profile b such that $U_i(p, I, b) \geqslant U_i(p, I, (b_i', b_{-i}))$ for all strategy b_i' and all buyer i. Let $\mathcal{E}(p, I)$ be the set of equilibria for (p, I). Because actions and signals are finite, the set of equilibria is always non-empty.

Let $R(p, I, b)$ be the expected revenue at an equilibrium b:

$$R(p, I, b) \equiv \int_{v,s,a} \sum_i a_i r(\Sigma a) p(\Sigma a) \left(\prod_i b_i(a_i|s_i) \right) \sigma(dv, ds).$$

Define the *revenue guarantee* of the market order mechanism with pricing rule p under the prior μ as the infimum expected revenue over all information structures $I \in \mathcal{I}(\mu)$ and all equilibria $b \in \mathcal{E}(p, I)$:

$$\underline{R}(p, \mu) \equiv \inf_{I \in \mathcal{I}(\mu)} \inf_{b \in \mathcal{E}(p,I)} R(p, I, b).$$

The revenue guarantee is also the minimum expected revenue across all *Bayes correlated equilibria* (see discussions in, e.g., Bergemann and Morris, 2016; Bergemann, Brooks, and Morris, 2017; Brooks and Du, 2021).

The main goal of this paper is to characterize the possible equilibrium welfare outcomes of market order mechanisms. Of particular interest is whether the revenue guarantee is close to the efficient surplus, which we regard as a measure of how competitive is the outcome. For some of our results, we will consider sequences of economies where N goes to infinity, and K, μ, and p may vary with N, but the range of possible values V will be held fixed.

3 Revenue Guarantees in Large Markets

3.1 Motivating examples

We now analyze what happens to expected revenue as the market grows large, depending on the sequence of pricing rules. Before describing our main results, we will illustrate what might happen under two natural candidates for the pricing rule. These examples will serve to illustrate forces that might induce non-competitive outcomes even when N is large and will serve to motivate the pricing rules that we propose. For motivating examples, we assume $K = 1$ and $V = [0, 1]$ for simplicity.

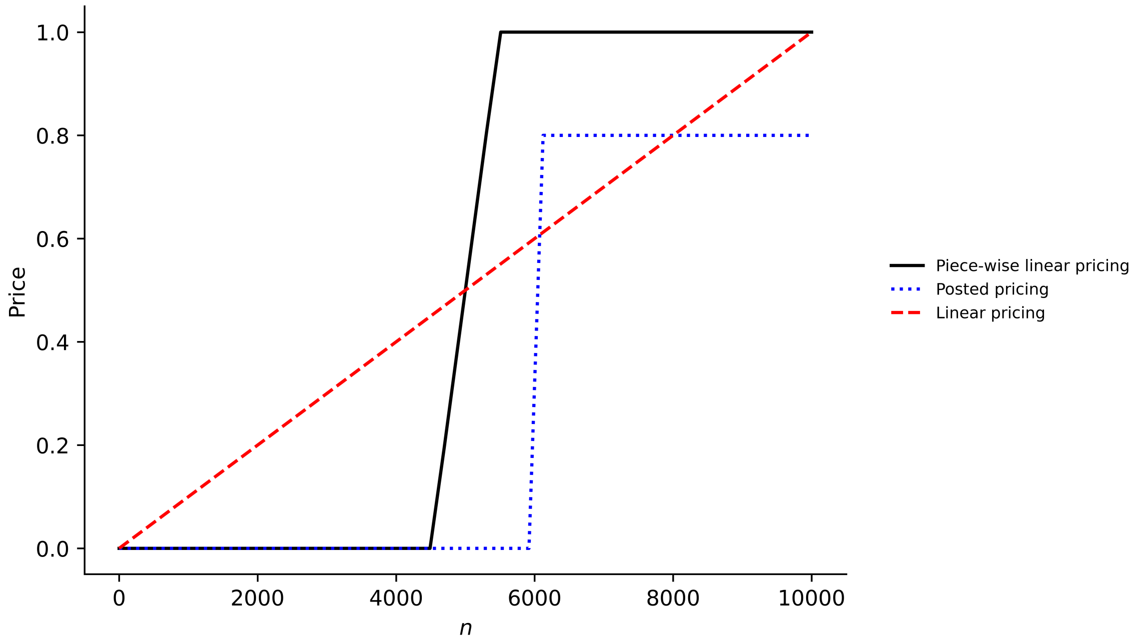

Figure 1: Pricing rules.

3.1.1 Posted Price

The first candidate is a class of "posted price" rules:

$$p(n) = \begin{cases} 0 & n \leqslant \widehat{n}, \\ \pi & n > \widehat{n}, \end{cases}$$

where $\widehat{n}$ is a participation cutoff and $\pi \in [0,1]$ is a posted price. As an example, this rule is depicted in Figure 1 in blue, with $\widehat{n} = 6000$ and $N = 10000$. It can be viewed as a generalization of the conventional posted price mechanism, which is obtained when $\widehat{n} = 0$.

Taking this pricing rule as given, we now exhibit an information structure for which expected revenue is bounded away from the expected value, no matter how large is the market. First, consider the case where $\pi < 1$. Suppose $v \in \{0,1\}$, both values equally likely. Furthermore, assume that the buyers have full information, meaning that for all i, $s_i = 1$ when $v = 1$ and $s_i = 0$ when $v = 0$. An equilibrium is that all buyers place orders if and only if $v = 1$. Therefore, regardless of N, revenue is $\pi/2$, which is strictly below the expected value of $1/2$, no matter how large is the market.

Now consider the case where $\pi = 1$. Suppose the information structure is such that only $\lfloor \widehat{n} \rfloor$ buyers have a full information about the value, and the rest of the buyers have no information beyond the prior distribution. Then it is an equilibrium for the $\lfloor \widehat{n} \rfloor$ buyers with full information to place orders if $v = 1$ and no orders otherwise, and the buyers with no information never place orders. Under such an equilibrium, the aggregate order is always just below $\widehat{n}$, and hence the market price is 0.

The reason for the low revenue in these cases is that the pricing rule has a jump from 0 to π, where the size of the jump is invariant with the number of buyers. The limited range of possible prices precludes the "discovery" of prices that are close to the expected value of marginal buyers who are not placing orders. This suggests that for a market order mechanism to induce a competitive outcome, price jumps must be negligible.

3.1.2 Linear Pricing

A natural pricing rule for which price increments would be small when the market is large is the linear rule $p(n) = n/N$. For comparison, this rule is also depicted in Figure 1 in red (with $N = 10000$).

Let us construct an information structure and equilibrium under the linear rule where expected revenue is far from the efficient surplus even when N is large. Again, suppose that $v \in \{0, 1\}$ and both are equally likely. Suppose N is even and let $S_i = \{0, 1, u\}$. These signals correspond to learning that the value is 0 or 1, or receiving an "uninformative" signal. If $v = 1$, then exactly $N/2$ of the buyers (uniformly drawn from the set of all buyers) observe the uninformative signal $s_i = u$, and the other $N/2$ buyers observe the perfectly informative signal $s_i = 1$. Likewise, if $v = 0$, then exactly $N/2$ buyers (uniformly drawn from the set of all buyers) observe the uninformative signal $s_i = u$, and the other $N/2$ observe the perfectly informative signal $s_i = 0$.

We claim that for this information structure, it is an equilibrium for the buyers to place orders if $s_i = 1$ and to not place orders otherwise. The equilibrium constraints for $s_i = 1$ and $s_i = 0$ are trivial, because under the proposed strategies, the price for the $s_i = 1$ types is just $1/2$, so they strictly prefer to place orders, whereas for the $s_i = 0$ types the price is 0, but the value is zero too, so they are happy to not place an order. For the uninformed $s_i = u$ types, the payoff from placing an order is

$$\frac{1}{2}\left(1 - \left(\frac{1}{2} + \frac{1}{N}\right)\right)\frac{1}{\frac{N}{2}+1} + \frac{1}{2}\left(0 - \frac{1}{N}\right) < \frac{1}{4}\frac{2}{N} - \frac{1}{2}\frac{1}{N} = 0.$$

As this payoff is non-positive, not placing an order is optimal for the $s_i = u$ types in equilibrium.

In this equilibrium, the price is positive only if $v = 1$, but the price is $1/2 < 1$, so the expected price is bounded away from the expected value regardless of N. In effect, there is a winner's curse that keeps the $s_i = u$ types from placing orders. Were an uninformed buyer to place an order, they would win with probability 1 when $v = 0$ and obtain a net payoff of $-1/N$, but they would only win with probability $2/(N + 2)$ when $v = 1$ and obtain a net payoff of $1/2 - 1/N$ conditional on winning. The net payoff is negative. In contrast, the rules we propose in the next section force the equilibrium participation rate to be in a narrow window with high probability. This effectively shuts down any updating about the value from the fact that one is allocated the good, and thereby precludes a winner's curse.

We note that while a winner's curse plays a prominent role in this example, Section 4.2 gives an example with inefficiently low sales even if the good does not need to be rationed. The more fundamental issue is that in these examples, the fraction of the population placing orders differs significantly when the value is high compared with when the value is low. Thus, the act of placing an order is itself informative about the value. This supports outcomes in which the value conditional on not placing orders is significantly below the value conditional on placing an order. When the market price is closer to the lower of these values, sales can be inefficiently low.

3.2 Sufficient Conditions for Competitive Outcomes

We now present our main result, which requires a few definitions.

The *price impact* of a pricing rule p is $\gamma = \max_n |p(n+1) - p(n)|$. Note that γ may vary with N, as p depends on N.

We say that $\underline{p} \in \mathbb{R}$ is an *admissible low price* if $\underline{v} = 0$ then $\underline{p} = 0$, and otherwise $\underline{p} < \underline{v}$.

A *window of price discovery* for a pricing rule p is a triple $(\underline{p}, x, \epsilon) \in \mathbb{R}^3$ with the following properties: (i) $\underline{p}$ is an admissible low price and (ii) if $n/N \geqslant x + \epsilon$ then $p(n) \geqslant \overline{v}$, and if $n/N \leqslant x - \epsilon$ then $p(n) \leqslant \underline{p}$. In other words, price discovery must occur in the window $[N(x - \epsilon), N(x + \epsilon)]$.

Theorem 1. *Fix an admissible low price $\underline{p}$ and $x \in (0, 1)$. Then there exist constants A, B, and C with the following property: For any N, K, μ, $\epsilon < x/2$, and pricing rule p with price impact γ and window of price discovery $(\underline{p}, x, \epsilon)$,*

$$\underline{R}(p, \mu) \geqslant \min\{K, Nx\} \left(\int_v v\mu(dv) - A\epsilon - B\gamma - C/N \right).$$

Thus, when ϵ and γ are small and N is large, revenue is approximately what it would be if $\min\{K, Nx\}$ units were sold at a price close to the ex ante expected value.

We can formalize this limit as follows. Fix a sequence of pricing rules (p_N) with associated price impacts γ_N. We say that the sequence has *vanishing price impact* if $\gamma_N \to 0$. We say that the sequence is *asymptotically inelastic (at x)* if there is a corresponding sequence

$(\underline{p}_N, x_N, \epsilon_N)$ of windows of price discovery that converge to $(\underline{p}, x, 0)$, where $\underline{p}$ is an admissible low price.

Note that we do not require the pricing rule p to be weakly increasing: we may allow for small decreases in p, as long as those decreases are negligible when $N \to \infty$.

A straightforward consequence of Theorem 1 is the following:

Corollary 1. *Suppose that there is a sequence of economies with N buyers, K_N units for sale, and priors $\mu_N \in \Delta([\underline{v}, \overline{v}])$. Let (p_N) be an associated sequence of pricing rules that has vanishing price impact and is asymptotically inelastic at $x \in (0, 1)$. Then*

$$\lim_{N \to \infty} \left(\frac{\underline{R}(p_N, \mu_N)}{\min\{K_N, Nx\}} - \int_v v \mu_N(dv) \right) = 0.$$

Thus, under the hypotheses of Corollary 1, market order mechanisms will asymptotically sell approximately $\min\{K_N, Nx\}$ units, and at a price that is equal to the value on average. In particular, if $K_N \leqslant \kappa N$ for all N for some $\kappa \in (0, 1)$, then by setting $x \geqslant \kappa$ the market order mechanisms' revenue guarantees are asymptotically equal to the efficient surplus.

Moreover, for Corollary 1, none of μ_N, K_N, or p_N are assumed to converge to any limit. Revenue per unit sold converges to the expected value, even if the distribution and pricing rules do not converge.

Returning to our motivating examples, notice that the posted price has a sudden price jump and a positive price impact that does not vary with N. The linear rule, on the other hand, has a wide window, with a width $\epsilon = 1/2$, regardless of N. Thus, neither of these sequences of pricing rules exhibit both vanishing price impact and asymptotic inelasticity.

For an example of pricing rules that *do* satisfy the hypotheses of Corollary 1, consider the following piecewise-linear pricing rules:

$$p(n) = \begin{cases} \underline{p} & \text{if } n \leqslant N(x - \epsilon); \\ \underline{p} + (\overline{v} - \underline{p}) \frac{n - N(x-\epsilon)}{2N\epsilon} & \text{if } N(x - \epsilon) < n \leqslant N(x + \epsilon); \\ \overline{v} & \text{if } n > N(x + \epsilon), \end{cases} \tag{1}$$

where $\underline{p}$ is an admissible low price. An example is depicted in Figure 1 in black, with $\underline{p} = 0$, $\overline{v} = 1$, $x = 0.5$, and $\epsilon = \frac{5}{\sqrt{N}}$. For illustration purposes, in Figure 1 we let $N = 10000$ so $\epsilon = 0.05$, but as N goes large, ϵ goes to 0 as desired. The piecewise-linear pricing rule is of independent interest, as we will explain shortly. Note that with pricing rules of this form, the price impact is exactly $\gamma = \frac{\overline{v} - \underline{p}}{2N\epsilon}$, and is equal to the increment in the price in the window $n/N \in [x - \epsilon, x + \epsilon]$.

Consider a sequence of pricing rules of this form, parameterized by windows $(\underline{p}, x, \epsilon_N)$. The sequence is asymptotically inelastic as long as ϵ_N converges to zero as N goes to infinity. On the other hand, the sequence has vanishing price impact if and only if $N\epsilon_N \to \infty$. Thus, for the hypotheses of Corollary 1 to hold, it is necessary that ϵ_N converge to zero but not too quickly. Thus, as a further corollary of Theorem 1, we have the following result:

Corollary 2. *Fix x and an admissible low price $\underline{p}$. Then there exist constants A, B, and C, such that for all piecewise-linear pricing rules of the form* (1) *and μ,*

$$\underline{R}(p, \mu) \geqslant \min\{K, Nx\} \left(\int_v v\mu(dv) - A\epsilon - B/(N\epsilon) - C/N \right). \tag{2}$$

The full proof of Theorem 1 is in Appendix A. To expose the logic underlying Theorem 1, in the section we present a direct proof of Corollary 2 for the special case where $\underline{p} = 0$. Afterwards, we will discuss the additional steps and ideas that are needed to prove Theorem 1.

Consistent with our previous discussion, the error bound (2) demonstrates that there are tradeoffs in choosing ϵ. In particular, making ϵ smaller reduces uncertainty about aggregate demand. However, the smaller is ϵ, the larger is the price impact when another buyer places an order. In fact, examining (2), it is clear that the optimal balance between these two forces is achieved when ϵ is on the order of $1/\sqrt{N}$, in which case the window and the price impact vanish at the same rate.

The convergence rate of $1/\sqrt{N}$ is in general unimprovable, since it is the rate given by the guarantee-maximizing mechanisms of Brooks and Du (2021) (i.e., proportional auctions) for a fixed distribution of the common value and a single unit of the good.[3] The $1/\sqrt{N}$ convergence rate to the efficient surplus is significantly better than the $1/\log(N)$ rate for the exponential price auction in Du (2018).[4] Moreover, market order mechanisms achieve the same rate even when multiple units are for sale, and as we will subsequently show, they work also if there is idiosyncratic components to values or multiunit demand. All of these cases were not covered in the prior literature.[5]

As an illustration, in Figure 2, we have plotted the revenue guarantees of the market order mechanisms with pricing rules in (1) for a setting in which $v \sim U[0, 1]$, $K = 1$, $x = 1/2$, and $\epsilon_N = 1/\sqrt{N}$. For comparison, we have also plotted the revenue guarantees of the optimal proportional auction of Brooks and Du (2021), the exponential-price auction of Du (2018),

[3] It is interesting to note that the market order mechanism can be viewed as a kind of "restricted" proportional auction, where bids are only allowed in $\{0, 1\}$.

[4] Du (2018) proves that $1/\log(N)$ is a lower bound on the rate of convergence for the exponential-price auction. The true rate could be higher.

[5] It is an open question whether $1/\sqrt{N}$ is the optimal rate for $K > 1$.

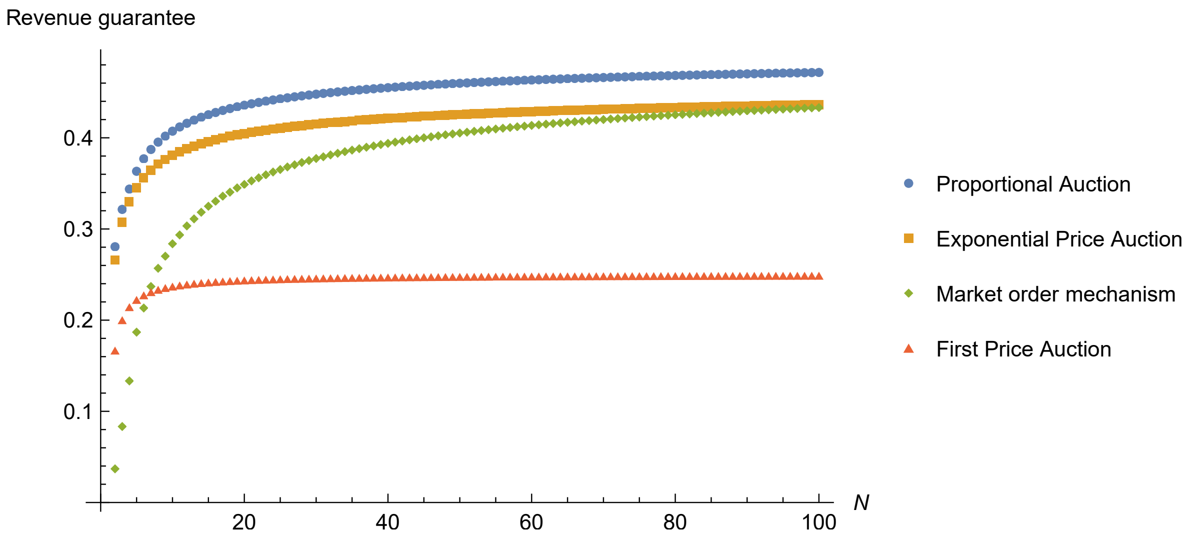

Figure 2: Comparison of revenue guarantees.

and the first-price auction. As we can see, even for moderate values of N, the market order mechanism outperforms the first-price auction, although it is still outperformed by the exponential-price auction. Around $N = 100$, the market order mechanism overtakes the exponential-price auction. Although it is still dominated by proportional auctions (as it must be), the gap is reduced to about 20% of the efficient surplus. As Theorem 1 shows, this gap must go to zero as N goes to infinity, at a rate of $1/\sqrt{N}$, as the revenue guarantees of both proportional auctions and market order mechanism converge to the efficient surplus.

3.3 Proof of Corollary 2 when $\underline{p} = 0$

The proof relies on methodology that was previously developed in Du (2018), Brooks and Du (2021), and related work. In particular, Brooks and Du (2024) present a general theory for informationally-robust mechanism design. For any mechanism, a lower bound on performance can be computed using an object known as the *strategic virtual objective*. In the context of market order mechanisms, the strategic virtual objective depends on a pair of parameters $\alpha \geqslant 0$ and $\beta \geqslant 0$, and is defined as

$$\lambda(v, n) = p(n) \min\{K, n\} + \alpha(N - n)r(n + 1)(v - p(n + 1)) - \beta nr(n)(v - p(n)). \tag{3}$$

The strategic virtual objective is essentially the objective in a Lagrangian for minimizing expected revenue subject to the constraints that in equilibrium, conditional on the action that they are taking, buyers must prefer that action to any alternative. These are referred to as *obedience constraints*. Importantly, for market order mechanisms, there are only two obedience constraints, which correspond to placing an order when one would have not done so, and not placing an order when one would have ordered. We have attached multipliers α and β to these constraints, respectively.[6] The expression (3) is, conditional on the value and the number of orders, a weighted sum of revenue; the multiplier-weighted change in payoff from ordering when would not have ordered; and the multiplier-weighted change in payoff from not ordering when one would have ordered.

The following result is established in the proof of Theorem 1 of Brooks and Du (2024):

Lemma 1. *The revenue guarantee of a market order mechanism is at least*

$$\int_v \min_n \lambda(v,n)\mu(dv). \tag{4}$$

For the sake of completeness, we will give a more detailed sketch of the logic behind the lower bound (4). In any information structure and equilibrium, there is some induced joint distribution $\sigma(n,v)$ of the number of buyers who place orders and the value, where the marginal of σ on v is μ. The resulting revenue is

$$\int_{n,v} p(n)\min\{K,n\}\sigma(dv,dn).$$

Now, buyers have the option to not place an order instead and secure a payoff of zero. As a result, the average utility of buyers who place orders must be non-negative:

$$\int_{n,v} nr(n)(v-p(n))\sigma(dv,dn) \geqslant 0. \tag{5}$$

At the same time, if n buyers are placing orders, there are $N-n$ buyers who are not. If one of these buyers were to instead place an order, they would have received a payoff of $r(n+1)(v-p(n+1))$. Since these buyers prefer to sit out and receive a payoff of zero, it must be that the expected counterfactual payoff is non-positive:

$$\int_{n,v} (N-n)r(n+1)(v-p(n+1))\sigma(dv,dn) \leqslant 0, \tag{6}$$

otherwise, some buyer who does not place an order in equilibrium, for some signal realization, must have a positive expectation of the payoff from placing an order. As a result, we can obtain a lower bound on revenue by taking expected revenue, subtracting the left-hand side of (5), and adding the right-hand side of (6) (where these extra terms are multiplied by

[6] In Du (2018), Brooks and Du (2021), and Brooks and Du (2024) the agents have many actions, which can be ordered. The strategic virtual objective only has non-trivial multipliers on the local obedience constraints.

non-negative weights). This is equivalent to the assertion that for any $\alpha \geqslant 0$ and $\beta \geqslant 0$, in any information structure and equilibrium, expected revenue is at least

$$\int_{n,v} \lambda(v,n)\sigma(dv,dn).$$

This expression is in turn weakly larger than what we obtain by, for each v, replacing $\lambda(v,n)$ with the minimum of $\lambda(v,\cdot)$, which is precisely (4).

We now have all of the pieces in place to prove Corollary 2.

Proof of Corollary 2. We use the lower bound from Lemma 1, with $\beta = 0$ and $\alpha = x/(1-x)$.

We consider three cases, depending on which of the piecewise linear segments of p the participation rate lies in:

Case 1: $n \leqslant N(x-\epsilon) - 1$. In this case, $p(n) = p(n+1) = 0$, and hence

$$\lambda(v,n) = \frac{N-n}{N(1-x)} \min\left\{K\frac{Nx}{n+1}, Nx\right\} v,$$

which is clearly decreasing in n. A lower bound is therefore obtained by setting $n = Nx$:

$$\begin{aligned}\lambda(v,n) \geqslant \min\left\{K\frac{Nx}{Nx+1}, Nx\right\} v &\geqslant \min\left\{K, Nx\right\}\left[v - \left(1 - \frac{Nx}{Nx+1}\right)\overline{v}\right] \\ &\geqslant \min\left\{K, Nx\right\}\left[v - \frac{1}{Nx}\overline{v}\right].\end{aligned}$$

Let $C = \overline{v}/x$.

Case 2: $n \geqslant N(x+\epsilon)$. In this case, $p(n) = p(n+1) = \overline{v}$, and hence

$$\lambda(v,n) = \min\{K,n\}\overline{v} + \frac{N-n}{N(1-x)} \min\left\{K\frac{Nx}{n+1}, Nx\right\}(v - \overline{v}),$$

which is increasing in n. We again obtain a lower bound by setting $n = Nx$:

$$\begin{aligned}\lambda(v,n) &\geqslant \min\{K,Nx\}\overline{v} + \min\left\{K\frac{Nx}{Nx+1}, Nx\right\}(v-\overline{v}) \\ &\geqslant \min\{K,Nx\}v,\end{aligned}$$

since $Nx/(Nx+1) \leqslant 1$.

Case 3: $n \in [N(x-\epsilon)-1, N(x+\epsilon)]$. Hence, $|p(n+1)-p(n)| \leqslant 1/(N\epsilon)$, and so

$$\lambda(v,n) \geqslant \min\{K,n\}p(n) + \frac{N-n}{N(1-x)}\min\left\{K\frac{Nx}{n+1}, Nx\right\}\left(v - p(n) - \frac{1}{N\epsilon}\right).$$

Now, $n/Nx \in [1-\epsilon/x, 1+\epsilon/x]$, so $\min\{K,n\}/\min\{K,Nx\} \in [1-\epsilon/x, 1+\epsilon/x]$ as well.[7] Similarly, we have that

$$\frac{N-n}{N(1-x)}\min\left\{K\frac{Nx}{n+1}, Nx\right\} \geqslant \min\left\{K\frac{1-\frac{\epsilon}{1-x}}{1+\frac{\epsilon}{x}+\frac{1}{N}}, Nx\left(1-\frac{\epsilon}{1-x}\right)\right\} \geqslant \min\{K,Nx\}\left(1-\frac{\epsilon}{1-x}\right)$$

and also

$$\frac{N-n}{N(1-x)}\min\left\{K\frac{Nx}{n+1}, Nx\right\} \leqslant \min\{K,Nx\}\frac{1+\frac{\epsilon}{1-x}}{1-\frac{\epsilon}{x}+\frac{1}{N}} \leqslant \min\{K,Nx\}\left(1+\frac{\epsilon}{1-x}\right).$$

(The last inequality uses that N is sufficiently large that $N\epsilon \geqslant x$, so the denominator in the center term is greater than 1.) Putting all of this together, and using $p \leqslant \overline{v}$, we have

$$\begin{aligned}\lambda(v,n) &\geqslant \min\{K,Nx\}\left[p(n) - \frac{\epsilon}{x}\overline{v} + v - p(n) - \overline{v}\frac{\epsilon}{1-x} - \frac{1}{N\epsilon N}\left(1+\frac{\epsilon}{1-x}\right)\right] \\ &= \min\{K,Nx\}\left[v - \epsilon\overline{v}\left(\frac{1}{x}+\frac{1}{1-x}\right) - \frac{2}{N\epsilon}\right].\end{aligned}$$

Hence, we can take

$$A = \overline{v}\left(\frac{1}{x}+\frac{1}{1-x}\right)$$

and $B = 2$.

Then clearly, the lower bound (2) is below the lower bounds that we derived in each case. □

This indirect approach to proving Corollary 2 sidesteps the issue of what would actually happen in equilibrium. This obviously depends on the particular form of information. However, the proof of Theorem 1 shows that if the participation rate is not in the band $[x-\epsilon, x+\epsilon]$, then the lower bound on revenue would be higher than the value. As a result, in equilibrium with a large market, the probability that the participation rate is outside $[x-\epsilon, x+\epsilon]$ must be close to zero, and the economy spends most of its time with intermediate prices. Since there is very little uncertainty about the participation rate, on the order of ϵ, regardless of

[7] If the minimum in both expressions is K, then the ratio is 1. If the ratio is n/K, then $N(x-\epsilon) \leqslant n \leqslant K \leqslant Nx$, so $1-\epsilon/x \leqslant K/(Nx) \leqslant 1$. The other cases are similar.

the underlying value, there must be very little information about the value contained in the decision to place an order. In particular, difference in expected values between buyers who place orders and those who don't must be of order ϵ. And since the price impact is at most $\delta = 1/(N\epsilon)$, it must be that the price is within δ of these expectations. When ϵ and δ are both small, then the equilibrium price must be close to the expected value.

The proof of Theorem 1 in the appendix for the general case is more complicated in two ways. First, we make fewer assumptions about the shape of p, and in the analogues of Cases 1 and 2, it need not be that λ is monotone in n. However, it is still the case that λ is minimized when n is close to Nx.

Second, the argument in Case 1 relied on the fact that the price is *exactly* zero when n is sufficiently below Nx. But Corollary 2 remains true if $\underline{p} \in (0, \underline{v})$. This generalization is substantive, because a monopoly seller who is convinced that the good is valuable may be uncomfortable with a rule that in principle could give away the good for free (which might happen with significant probability if the market is not in equilibrium). But with $\underline{p} > 0$, the given multipliers on obedience constraints are not optimal, and the resulting lower bound would be too low to prove that expected revenue per unit sold converges to the expected value. In particular, if $p(n) = \underline{p} > 0$ when $n < N(x - \epsilon)$, then in Case 1, the strategic virtual objective would be

$$\lambda(v, n) = \min\{K, n\}\underline{p} + \frac{N - n}{N(1 - x)} \min\left\{K\frac{Nx}{n + 1}, Nx\right\}(v - \underline{p}),$$

which is not necessarily decreasing in n. In fact, when $\underline{p} > 0$, both obedience constraints bind and the optimal multipliers are both strictly positive. It is this more complicated Lagrangian that we work with in the proof of Theorem 1.

4 Welfare Guarantees in Large Decentralized Markets

4.1 Interpreting p as a Supply Curve

In Section 3, we took the pricing rule as given and characterized revenue and buyer surplus. But what about welfare? One interpretation is that the pricing rule is chosen by a monopoly seller who has zero production cost. In this case, welfare is simply equal to revenue from buyer surplus. Our results show that in large markets, the seller can achieve an efficient outcome and extract nearly all of the surplus, regardless of the information structure and equilibrium, using appropriately chosen market order mechanisms.

Another interpretation is that the pricing rule represents the supply curve in a competitive market. In this case, the welfare implications of the model are more subtle. We now study welfare more formally in the case of a large decentralized market.

In particular, suppose that there are $M \geqslant 2$ producers of the good. Seller m can supply k units of the good at cost $C_m(k)$. Let $c_m(k) = C_m(k) - C_m(k-1)$ be the marginal cost function of producer m. We assume that c_m is non-decreasing for all m. In this richer model, we assume the following sequential structure for how the market clears: First, buyers place their orders, as before. Then, after seeing which buyers placed orders, each producer posts a price p_m. The buyers who placed orders then choose from which producer to purchase. Finally, orders are fulfilled, and the buyers pay the producer that they patronize. Note that we assume that producers can make as many units as ordered, possibly at very high cost, so that we are also implicitly assuming that $K = N$.

The subgame after the buyers have placed orders is a standard model of Bertrand competition. All equilibria of this model have the property that when n orders have been placed, the producers will compete the price down so that it is between the nth and $n+1$th lowest marginal costs. In particular, we can define the aggregate cost function:

$$C(n) = \min\{C_1(n_1) + \cdots + C_M(n_M) | n_1 + \cdots + n_M = n\},$$

and the aggregate marginal cost $c(n) = C(n) - C(n-1)$. Then the equilibrium price is in the range $[c(n), c(n+1)]$, and the lowest marginal cost producers fill the orders. We focus on the equilibrium in which the price is $p(n) = c(n)$, the marginal cost to produce the last unit.

Hence, given information I and strategies b, total welfare is

$$W(p, I, b) \equiv \int_{v,s,a} \left((\Sigma a)\, v - \sum_{m \leqslant \Sigma a} p(m) \right) \prod_i b_i(a_i|s_i) \sigma(dv, ds),$$

i.e., the value of the units sold, less the production cost. We let $\underline{W}(p)$ be the *welfare guarantee* given p and μ:

$$\underline{W}(p, \mu) \equiv \inf_{I \in \mathcal{I}(\mu)} \inf_{b \in \mathcal{E}(p,I)} W(p, I, b).$$

If the buyers had no information, then there would be common knowledge that the expected value is simply $\int_v v\mu(dv)$. The buyers would simply place orders until the price is competed up to that level. The resulting ex ante social welfare under no information is then

$$W^*(\mu) \equiv \max_n \left[n \int_v v\mu(dv) - \sum_{m \leqslant n} p(m) \right]. \tag{7}$$

4.2 An Example

W^* is the highest level that we could hope to guarantee for welfare, since it is always possible that buyers have no information. However, it is in general possible for welfare to be even lower than W^*, as the following example shows.

Suppose that $v \in \{0, 1\}$, both equally likely, and the pricing rule is $p(n) = (2/3)\mathbb{I}_{n>N/2}$. (We assume for this example that N is even.) So, the good is costless to produce to cover half the population, but above that point, the marginal cost jumps up to $2/3$. Under no information, the efficient outcome is for exactly half of the units to be sold, attaining a welfare of $N/4$. Now, consider the following information structure: When the value is 0, with probability $1/3$, exactly half of the buyers (chosen at random) receive a signal that tells them not to buy. Otherwise, all of the buyers receive a signal telling them to buy. Similarly, when $v = 1$, with probability $2/3$, exactly half of the buyers are told to not buy, and otherwise they are all told to buy. In equilibrium, the buyers follow these recommendations.

Now, conditional on not buying, exactly $N/2$ buyers are buying, so switching to buying would cause the price to jump up to $2/3$. The expected value conditional on not buying is $2/3$ (because not buying is twice as likely when $v = 1$) so that the payoff from switching to buy is zero. Moreover, expected consumer surplus per capita from following the equilibrium strategy is

$$\frac{1}{2}\frac{2}{3}\left(0 - \frac{2}{3}\right) + \frac{1}{2}\left[\frac{1}{2}\frac{2}{3}(1-0) + \frac{1}{3}\left(1 - \frac{2}{3}\right)\right] = -\frac{2}{9} + \frac{1}{6} + \frac{1}{18} = 0.$$

Hence, buyers who place orders do not wish to rescind them. Finally, total surplus in this example is

$$\frac{1}{2}\left[\frac{1}{3}\left(\frac{N}{2}0 - \frac{N}{2}0\right) + \frac{2}{3}\left(N0 - \frac{N}{2}0 - \frac{N}{2}\frac{2}{3}\right)\right] + \frac{1}{2}\left[\frac{2}{3}\left(\frac{N}{2}1 - \frac{N}{2}0\right) + \frac{1}{3}\left(N1 - \frac{N}{2}0 - \frac{N}{2}\frac{2}{3}\right)\right]$$
$$= N\left(-\frac{1}{9} + \frac{1}{6} + \frac{1}{9}\right) = \frac{N}{6} < \frac{N}{4}.$$

Why in this example is social welfare lower than under no information? Comparing the two outcomes, the number of purchases is quite a bit higher, at $3N/4$ compared to $N/2$. However, the expected value conditional on a purchase is lower. Indeed, as we saw in our previous examples, when price impact is high, it is possible that the expected value of non-buyers is higher than the expected value of buyers. This is precisely what happens in this example; the expected value of non-buyers is 2/3, which is what the price would jump up to if any of them were to purchase. In addition, while expected value conditional on a purchase is lower than under no information, the average production cost is higher, because now with probability 1/2 all buyers purchase.

4.3 Welfare Guarantees and Price Impact

The inefficiency in the preceding example relies on the fact that price impact is large. In fact, as the next result shows, when price impact is small, equilibrium welfare cannot be far below the no information benchmark:

Theorem 2. *Suppose that price impact is at most γ. Then*

$$\underline{W}(p,\mu) \geqslant W^*(\mu) - N\gamma.$$

Proof. Let n^* be the maximizer of (7). Define the strategic virtual objective[8]

$$\lambda(v,n) = nv - \sum_{m\leqslant n} p(m) + \frac{n^*}{N}(N-n)(v-p(n+1)) - \left(1-\frac{n^*}{N}\right)n(v-p(n)).$$

Rearranging this expression, we have

$$\begin{aligned}\lambda(v,n) &= n^*v - \sum_{m\leqslant n} p(m) - \frac{n^*}{N}(N-n)p(n+1) + \left(1-\frac{n^*}{N}\right)np(n)\\ &\geqslant n^*v \underbrace{- \sum_{m\leqslant n} p(m) + (n-n^*)\,p(n)}_{\equiv J(n)} - \gamma n^*\frac{N-n}{N}.\end{aligned}$$

Now,

$$\begin{aligned}J(n+1)-J(n) &= (n+1-n^*)p(n+1) - (n-n^*)p(n) - p(n+1)\\ &= (n-n^*)(p(n+1)-p(n)).\end{aligned}$$

[8] Comparing with equation (3), the objective has changed from revenue to welfare, the multipliers are now $\alpha = \frac{n^*}{N}$, $\beta = 1-\frac{n^*}{N}$, and $r(n) = r(n+1) = 1$ since $K = N$.

Hence, J is single-troughed at $n = n^*$, and we have

$$\lambda(v, n) \geqslant n^* v - \sum_{m \leqslant n^*} p(m) - \gamma N$$

for every n.

Therefore, at any outcome $\sigma(dv, dn)$ induced by an equilibrium, we have (see the discussion following Lemma 1):

$$\int_{n,v} \left(nv - \sum_{m \leqslant n} p(m) \right) \sigma(dv, dn) \geqslant \int_{v,n} \lambda(v, n) \sigma(dv, dn) \geqslant \int_{v} \left(n^* v - \sum_{m \leqslant n^*} p(m) \right) \mu(dv) - \gamma N.$$

$\square$

Of course, it might be that buyers in fact have no information, in which case welfare can be at most W^*. We therefore have the following analogue of Corollary 1:

Corollary 3. *For any sequence of economies (N, μ_N, p_N) with vanishing price impact,*

$$\lim_{N \to \infty} \left(\frac{\underline{W}(p_N, \mu_N)}{N} - \frac{W^*(\mu_N)}{N} \right) = 0.$$

In words, when there is vanishing price impact, the social welfare guarantee per capita must converge to that which is obtained under no information.

In general, W^* is less than the ex post efficient surplus, which is

$$W^{**} = \int_{v} \max_{n} \left(nv - \sum_{m \leqslant n} p(m) \right) \mu(dv).$$

However, one special case where W^{**} and W^* coincide, in limit as N grows large, is when the sequence of prices functions/aggregate marginal cost curves (p_N) is asymptotically inelastic at some $x \in (0, 1)$. The reason is that in that limit, social efficiency only requires that $n \approx Nx$, which is achievable under no information. Hence, we have the following further corollary of Theorem 2:

Corollary 4. *For any sequence of economies (N, μ_N, p_N) that has vanishing price impact and is asymptotically inelastic, then*

$$\lim_{N \to \infty} \left(\frac{\underline{W}(p_N, \mu_N)}{N} - \frac{W^{**}(\mu_N)}{N} \right) = 0.$$

Note that, by Corollary 1, under the hypotheses of Corollary 4, we also have revenue per capita converging to the ex ante expected value. Hence, buyer surplus goes to zero, no matter the sequence of information structures and equilibria.

5 Heterogeneous Values

5.1 Revenue Guarantees in Large Markets

We now extend our results beyond the case of pure common values. Suppose the value has both common and private components, i.e., the value of buyer i is $v(\nu, \omega_i) \in [\underline{v}, \overline{v}]$, where $\nu \in \mathcal{V}$ is the common value component, and $\omega_i \in \Omega_i \subset \mathbb{R}$ is buyer i's private value component. As an example, ν could represent the resale value of the good, while ω_i is a private use value. For analytical simplicity, we suppose that $\mathcal{V}$ and the Ω_i are all finite sets. We further suppose that $v(\nu, \omega_i)$ is strictly increasing in ω_i. A leading example is the additively separable form $v(\nu, \omega_i) = \nu + \omega_i$. Versions of this model have previously been studied by Pesendorfer and Swinkels (2000), Jackson (2009), and McLean and Postlewaite (2023).

Given a pricing rule p, buyer i's utility in a market order mechanism is now $u_i = a_i r(\Sigma a)(v(\nu, \omega_i) - p(\Sigma a))$. Each buyer i observes his private value ω_i and also observes a signal s_i about the common value ν as well as others' private values ω_{-i}, as described by an information structure $I = (S, \sigma)$, where S_i is the set of signals for buyer i $S = \prod_i S_i$, $\Omega = \prod_i \Omega_i$, and $\sigma \in \Delta(\mathcal{V} \times \Omega \times S)$. As a result, a strategy for buyer i is now a mapping $b_i : \Omega_i \times S_i \to \Delta(A_i)$. Note that we allow for arbitrary correlation between ω_i and the signal about ν, e.g., a buyer with a high ω_i may receive a pessimistic signal s_i about ν, potentially presenting an obstacle to allocative efficiency (which depends only on ω).

Let $\mu = \operatorname{marg}_{\mathcal{V} \times \Omega} \sigma$. For a market order mechanism with pricing rule p, the revenue guarantee $\underline{R}(p, \mu)$ is defined, as before, as infimum expected revenue over all information structures with marginal μ and all equilibria.

We make the following assumption about μ_N as $N \to \infty$. Let F_N be the empirical cumulative distribution function (CDF) for $\omega \in \Omega$, i.e.,

$$F_N(z) = \frac{|\{i : \omega_i \leqslant z\}|}{N}$$

for $z \in \mathbb{R}$. We assume that there exists a CDF F such that

$$\lim_{N \to \infty} \frac{N}{K_N} \mu_N \left(\sup_{z \in \mathbb{R}} |F_N(z) - F(z)| > \delta_N \right) = 0 \tag{8}$$

for some sequence $\delta_N \to 0$.

Condition (8) says that the uncertainty about the empirical distribution of the private values vanishes as the market gets large. Moreover, each buyer knows with very high precision their quantile in that empirical distribution. For example, Condition (8) is satisfied if in μ_N the ω_i's are independently and identically drawn from F: the Dvoretzky–Kiefer–Wolfowitz inequality (Massart, 1990) states that

$$\mu_N\left(\sup_{z\in\mathbb{R}} |F_N(z) - F(z)| > \delta\right) \leqslant 2\exp(-2N\delta^2),$$

for every $\delta > 0$ and N. Thus to satisfy Condition (8) we can take $\delta_N = N^{-c}$ for any $c < 1/2$.

Let F^{-1} be the quantile function: $F^{-1}(r) = \inf\{z \in \mathbb{R} : F(z) \geqslant r\}$ for $r \in [0,1]$. Thus $F^{-1}(r)$ is the r-th percentile private value.

Theorem 3. *Suppose $K_N \to \infty$ as $N \to \infty$. Let (μ_N) be a sequence of priors satisfying Condition (8) for a CDF F, and let (p_N) be a sequence of pricing rules with $\underline{p} = 0$ that has vanishing price impact and is asymptotically inelastic at $x \in (0,1)$. Then we have*

$$\lim_{N\to\infty}\left(\frac{\underline{R}(p_N,\mu_N)}{\min\{K_N, Nx\}} - \int_{\nu,\omega} v(\nu, F^{-1}(1-x))\mu_N(d\nu, d\omega)\right) = 0. \tag{9}$$

The proof of Theorem 3 is essentially a reduction to the pure common value case. As the buyers are assumed to know their private values, there are now separate obedience constraints for each ω_i. Thus, the multipliers α and β on the obedience constraints (see the discussion on page 15) are now functions of ω_i, and the strategic virtual objective becomes (cf. equation (3))

$$\begin{aligned}\lambda(\nu,\omega,a) = p(\Sigma a)\min\{K,\Sigma a\} + \sum_i \Big(&\alpha_i(\omega_i)\mathbb{I}_{a_i=0} r(\Sigma a+1)(v(\nu,\omega_i) - p(\Sigma a+1)) \\ &- \beta_i(\omega_i)\mathbb{I}_{a_i=1} r(\Sigma a)(v(\nu,\omega_i) - p(\Sigma a))\Big),\end{aligned}$$

with

$$\underline{R}(p,\mu) \geqslant \int_{\nu,\omega} \min_a \lambda(\nu,\omega,a)\mu(d\nu,d\omega).$$

Let $y = F^{-1}(1-x)$. We set

$$\alpha_i(\omega_i) = \begin{cases} \overline{\alpha} & \omega_i \geqslant y, \\ 0 & \omega_i < y, \end{cases} \qquad \beta_i(\omega_i) = \begin{cases} 0 & \omega_i \geqslant y, \\ \overline{\alpha} & \omega_i < y, \end{cases}$$

where $\overline{\alpha} > 0$. Thus, the constraint for deviating from not buy to buy binds for the buyers with high private value and the constraint for deviating from buying to not buying binds for buyers with low private value. This is consistent with an outcome in which the good is ultimately allocated to the high private value buyers. With these choices of multipliers, we have

$$\begin{aligned}\lambda(\nu,\omega,a) \geqslant & p(n_1+n_2)\min\{K,n_1+n_2\} + \overline{\alpha}(N_1-n_1)(v(\nu,y)-p(n_1+n_2+1))r(n_1+n_2+1) \\ & - \overline{\alpha}n_2(v(\nu,y^-)-p(n_1+n_2))r(n_1+n_2) \\ \equiv & \underline{\lambda}(\nu,\omega,n_1,n_2),\end{aligned}$$

where y^- is the private value immediately below y, $N_1 = |\{i : \omega_i \geqslant y\}|$, $n_1 = |\{i : \omega_i \geqslant y, a_i = 1\}|$, and $n_2 = |\{i : \omega_i < y, a_i = 1\}|$. Thus it suffices to focus on $\min_{n_1,n_2} \underline{\lambda}(\nu,\omega,n_1,n_2)$. But since $v(\nu,y) > v(\nu,y^-)$, to minimize $\underline{\lambda}(\nu,\omega,n_1,n_2)$ while fixing n_1+n_2 we clearly want to set $n_2 = 0$. Thus $\underline{\lambda}(\nu,\omega,n_1,n_2)$ is reduced to the strategic virtual objective in (3) with $N = N_1$, $n = n_1$, $v = v(\nu,y)$, $\alpha = \overline{\alpha}$, and $\beta = 0$, and we proceed as in the proof of Theorem 1. In particular, condition (8) implies that when N is large the empirical distribution F_N is close to F with high probability, so N_1 is approximately $N(1-F(y^-))$. Additional detail of the proof can be found in Appendix A.

In the case where $K_N = \lfloor \kappa N \rfloor$ for some $\kappa \in (0,1)$, Theorem 3 implies that the market order mechanism with $x = \kappa$ always yields the competitive price which is the expected value of the marginal buyer who exhausts the supply.

In some cases the revenue guarantees in Theorem 3 are asymptotically optimal. Continue to assume that $K_N = \lfloor \kappa N \rfloor$ for some $\kappa \in (0,1]$. Also suppose that the sequence of priors μ_N satisfies condition (8) and that the common value ν has the fixed marginal distribution $\tilde{\mu}$. Finally, suppose

$$x\int_{\nu,\omega} v(\nu,F^{-1}(1-x))\tilde{\mu}(d\nu)$$

is a concave function of $x \in [0,\kappa]$.[9] Then solving[10]

$$\sup_{x\in(0,\kappa]} x\int_{\nu,\omega} v(\nu,F^{-1}(1-x))\tilde{\mu}(d\nu), \tag{10}$$

[9]This is the "regular" case where the virtual value from the private value is non-decreasing; see Bulow and Roberts (1989).

[10]We must use sup instead of max because F^{-1} is not continuous from the right.

gives revenue guarantees in (9) arbitrarily close to asymptotic optimality. To see this, consider the information structure in which each buyer i only observes his private value ω_i and has no information about the common value ν. Then the analysis of Myerson (1981) shows that as $N \to \infty$, the optimal revenue per capita under incentive compatible and individual rational mechanisms is precisely (10).

Theorem 3 is closely related to Pesendorfer and Swinkels (2000), who show that equilibrium price in the $(K+1)$-th price auction converges to the value of the marginal buyer (with the K-th highest value) in the environment with both common and private components in value. Pesendorfer and Swinkels (2000) prove this result for the symmetric and monotone equilibrium[11] in a specific information structure where buyers' signals are independently distributed conditional on the common value component; in contrast, our price convergence result holds for every information structure and equilibrium. Note that there are information structures and equilibria in which the price in the $(K+1)$-th price auction is bounded away from the common value in expectation as $N \to \infty$, e.g., the maximum signal information structure in Bergemann, Brooks, and Morris (2017) for $K = 1$. Thus, when N is large, the $(K+1)$-th price auction has a strictly inferior revenue guarantee compared to market order mechanisms, when the window of price discovery is small and centered around $x \approx K/N$.

McLean and Postlewaite (2023) also study a large market with buyers having both common and private components in value and a single good. They construct a two-stage mechanism where there is voting about the common value component in the first stage (which fully reveals the common value component), and the second stage is a second price auction. Similar to Pesendorfer and Swinkels (2000), McLean and Postlewaite (2023) rely on the buyers being symmetric and having independent signals conditional on the common value component. Our results rely on neither assumption. On the other hand, the result of McLean and Postlewaite (2023) are able to relax the common prior assumption by only assuming common knowledge among the buyers of lower and upper bounds on the precision of their signals.

5.2 Welfare Guarantees in Large Decentralized Markets

We next revisit the decentralized market model in Section 4 with common and private values. Given an information structure I and strategies b, total welfare is now

$$W(p, I, b) = \int_{\nu,\omega,s,a} \left((\Sigma a)\, v(\nu, \omega_i) - \sum_{m \leqslant \Sigma a} p(m) \right) \prod_i b_i(a_i | s_i, \omega_i) \sigma(d\nu, d\omega, ds),$$

[11] A caveat of their result is that Pesendorfer and Swinkels (2000) do not prove the existence of a monotone and symmetric equilibrium and simply characterize the implications of the equilibrium. Jackson (2009) shows that non-existence of equilibrium is a real concern in this setting. Since we work with finite type spaces and a finite mechanism, an equilibrium always exists.

i.e., the value of the units sold, less the production cost. As before, we define the welfare guarantee $\underline{W}(p,\mu)$ as the minimum expected total welfare over all information structures with marginal μ and all equilibria.

Ex ante social welfare under no information is

$$W^*(\mu) = \max_n \left[\int_{\nu,\omega} \sum_{m \leqslant n} v(\nu, \omega^{(m)}) \mu_N(d\nu, d\omega) - \sum_{m \leqslant n} p(m) \right], \tag{11}$$

where $\omega^{(m)}$ is the m-th highest value in ω.

As in the previous subsection, we consider a sequence of priors (μ_N) where the uncertainty about the empirical distribution of private values vanishes as the market gets large: there exists a CDF F such that

$$\lim_{N\to\infty} \mu_N \left(\sup_{z\in\mathbb{R}} |F_N(z) - F(z)| > \delta_N \right) = 0 \tag{12}$$

for some sequence $\delta_N \to 0$. (This is condition (8) for the special case of $K_N = N$.)

Theorem 4. *Let (μ_N) be a sequence of priors satisfying (12) for a CDF F, and let (p_N) be a sequence of supply functions with vanishing price impact. Then*

$$\lim_{N\to\infty} \left(\frac{\underline{W}(p_N,\mu_N)}{N} - \frac{W^*(\mu_N)}{N} \right) = 0.$$

The proof of Theorem 4 is in Appendix A. It largely follows the same argument as the proof of Theorem 2, with the additional structure on obedience constraints that we described after the statement of Theorem 3.

The ex post efficient surplus is

$$W^{**}(\mu) = \int_{\nu,\omega} \max_n \left(\sum_{m \leqslant n} v(\nu, \omega^{(m)}) \mu_N(d\nu, d\omega) - \sum_{m \leqslant n} p(m) \right).$$

As in the case of pure common values, we have the following corollary of Theorem 4:

Corollary 5. *Fix $\underline{v}$ and $\overline{v}$. Let (μ_N) be a sequence of priors satisfying Condition (12) for a CDF F, and let (p_N) be a sequence of supply functions that has vanishing price impact and is asymptotically inelastic. Then we have*

$$\lim_{N\to\infty} \left(\frac{\underline{W}(p_N,\mu_N)}{N} - \frac{W^{**}(\mu_N)}{N} \right) = 0.$$

6 Multi-Unit Demands

In many important markets, such as initial public offerings of shares of equity, buyers demand more than one unit, with different amounts being demanded by different buyers. In this section, we show how multi-unit demands can be incorporated into our analysis.

For simplicity, let us focus on the analysis of revenue guarantees with common values, which we considered with single-unit demand in Section 3. Now, there are K units of a good for sale and N buyers. Each buyer i, $1 \leqslant i \leqslant N$, has a common value v per unit of the good for the first m_i units, and zero value for any additional units. Suppose there is a constant $\overline{m}$ such that

$$m_i \leqslant \overline{m}$$

for all i. We will hold $\overline{m}$ fixed as we increase the number N of buyers.

The market order mechanism now has an action space $A_i = \{0, 1, \ldots, \overline{m}\}$, a rationing function $r : \{0, 1, \ldots, N\overline{m}\} \to [0, 1]$, and a pricing function $p : \{0, 1, \ldots, N\overline{m}\} \to [0, \mathcal{C}]$. Each $a_i \in A_i$ represents a market order from buyer i for a_i units of the good; they are fulfilled at a rate of $r(n) = \min(K/n, 1)$, where $n = \sum_i a_i$, so buyer i gets $a_i r(n)$ units at a price of $p(n)$ per unit.

Our main result for this section is the following:

Theorem 5. *Suppose $K_N = \lfloor \kappa N \rfloor$. Let $((m_i^N)_{1 \leqslant i \leqslant N})$ be a sequence of multi-unit demands satisfying $m_i^N \leqslant \overline{m}$ for each i and N, fix x such that $x < \frac{\sum_{i=1}^N m_i^N}{\overline{m} N}$ for each N, and let (p_N) be a sequence of pricing rules that has vanishing price impact and is asymptotically inelastic at x. Then we have*

$$\lim_{N \to \infty} \left(\frac{\underline{R}(p_N, \mu_N)}{\min(K_N, xN\overline{m})} - \int_v v \, \mu_N(dv) \right) = 0$$

for any sequence of priors (μ_N).

Theorem 5 implies that to obtain the optimal revenue guarantee in a large market, the seller does not need precise information about individual demands $(m_i)_{1 \leqslant i \leqslant N}$, but simply an upper bound on demand m_i and a lower bound on $\sum_{i=1}^N m_i$. As long as K is smaller than this lower bound, the seller can guarantee to sell out all K units at an expected price arbitrarily close to the ex ante expected value.

The proof of Theorem 5 proceeds by effectively reducing the analysis to the unit demand case. In particular, an equivalent way to represent the market order mechanism is the following: each buyer i has $\overline{m}$ "accounts" numbered 1 through $\overline{m}$; through each account buyer i can submit a market order for one unit of the good, which is rationed at $r(n)$ and has a price of $p(n)$, where n is the total number of market orders submitted by all accounts of all buyers. Moreover, for buyer i the goods acquired by his accounts 1 through m_i generate a value of v, while the goods acquired by accounts $m_i + 1$ through $\overline{m}$ generate zero value. With a slight abuse of notation, let $a_i = (a_{i,1}, a_{i,2})$, where $a_{i,1}$ is the number of market orders submitted by accounts 1 through m_i, and $a_{i,2}$ is the number of market orders submitted by accounts $m_i + 1$ through $\overline{m}$. Then buyer i's utility is

$$u_i(a_{i,1}, a_{i,2}, a_{-i}, v) = a_{i,1} r(\Sigma a)(v - p(\Sigma a)) - a_{i,2} r(\Sigma a) p(\Sigma a),$$

where $\Sigma a = \sum_i a_{i,1} + a_{i,2}$.

The strategic virtual objective of the market order mechanism is: for any v and $a = (a_{i,1}, a_{i,2})_{1 \leqslant i \leqslant N}$ such that $a_{i,1} \leqslant m_i$ and $a_{i,2} \leqslant \overline{m} - m_i$,

$$\begin{aligned}\lambda(v, a) = &\min\{\Sigma a, K\} p(\Sigma a) \\ &+ \sum_i \Big[\alpha(m_i - a_{i,1})(u_i(a_{i,1} + 1, a_{i,2}, a_{-i}, v) - u_i(a_{i,1}, a_{-i}, v)) \\ &\qquad + \beta a_{i,1}(u_i(a_{i,1} - 1, a_{i,2}, a_{-i}, v) - u_i(a_{i,1}, a_{i,2}, a_{-i}, v)) \\ &\qquad + \eta a_{i,2}(u_i(a_{i,1}, a_{i,2} - 1, a_{-i}, v) - u_i(a_{i,1}, a_{i,2}, a_{-i}, v))\Big].\end{aligned} \tag{13}$$

Equation (13) is the Lagrangian for minimizing revenue, subject to the subset of obedience constraints where buyers change their order on only one account at a time. As before, α (respectively, β) is the multiplier on the obedience constraints for deviating to not placing an order (respectively, to placing an order) with account j, $1 \leqslant j \leqslant m_i$; similarly, η is the multiplier on the obedience constraints deviating to placing an order with account j, $m_i < j \leqslant \overline{m}$. (We set the multiplier on the obedience constraints for deviating to not placing an order with account j, $m_i < j \leqslant \overline{m}$, to be zero.) Note that we are ignoring deviations that are coordinated across a buyer's accounts. Hence, we are not using the full implications of an equilibrium in the market order mechanism. However, as the following argument shows, the subset of constraints to which we have attached non-zero multipliers are sufficient for pinning down equilibrium revenue and buyer surplus, when the market is large.

Proceeding formally, note that

$$|r(\Sigma a + 1) - r(\Sigma a)| = \left|\min\left\{\frac{K}{\Sigma a + 1}, 1\right\} - \min\left\{\frac{K}{\Sigma a}, 1\right\}\right| \leqslant \frac{1}{K}.$$

Hence,

$$\begin{aligned}&|r(\Sigma a + 1)p(\Sigma a + 1) - r(\Sigma a)p(\Sigma a)| \\ &\leqslant |r(\Sigma a + 1)p(\Sigma a + 1) - r(\Sigma a + 1)p(\Sigma a)| + |r(\Sigma a + 1)p(\Sigma a) - r(\Sigma a)p(\Sigma a)|\end{aligned}$$

$$\leqslant \gamma + \mathcal{C}/K$$

where γ is the price impact for the pricing function p. These together imply that

$$\begin{aligned}
&u_i(a_{i,1}+1, a_{i,2}, a_{-i}, v) - u_i(a_{i,1}, a_{i,2}, a_{-i}, v) \\
&= r(\Sigma a + 1)(v - p(\Sigma a + 1)) + a_{i,1}(r(\Sigma a + 1)(v - p(\Sigma a + 1)) - r(\Sigma a)(v - p(\Sigma a))) \\
&\quad + a_{i,2}(-r(\Sigma a + 1)p(\Sigma a + 1) + r(\Sigma a)p(\Sigma a)) \\
&\geqslant r(\Sigma a)(v - p(\Sigma a)) - (\overline{m} + 1)(\overline{v}/K + \gamma + \mathcal{C}) \\
&= r(\Sigma a)(v - p(\Sigma a)) - \delta
\end{aligned}$$

where

$$\delta = (\overline{m} + 1)(\overline{v}/K + \gamma + \mathcal{C}).$$

By a similar derivation,

$$u_i(a_{i,1}-1, a_{i,2}, a_{-i}, v) - u_i(a_{i,1}, a_{i,2}, a_{-i}, v) \geqslant -r(\Sigma a)(v - p(\Sigma a)) - \delta,$$

and

$$u_i(a_{i,1}, a_{i,2}-1, a_{-i}, v) - u_i(a_{i,1}, a_{i,2}, a_{-i}, v) \geqslant r(\Sigma a)p(\Sigma a) - \delta.$$

Using these inequalities, we can bound (13) from below by:

$$\begin{aligned}
\lambda(v, a) \geqslant &\min\{n_1 + n_2, K\}p(n_1 + n_2) \\
&+ \alpha(\Sigma m - n_1)(r(n_1 + n_2)(v - p(n_1 + n_2)) - \delta) \\
&- \beta n_1(r(n_1 + n_2)(v - p(n_1 + n_2)) + \delta) \\
&+ \eta n_2(r(n_1 + n_2)p(n_1 + n_2) - \delta) \\
\equiv &\underline{\lambda}(v, n),
\end{aligned} \tag{14}$$

where $n_1 = \sum_i a_{i,1}$ and $n_2 = \sum_i a_{i,2}$.

As long as we take η to be larger than $\alpha + \beta$, then the coefficient on n_2 in (14) is larger than the coefficient of n_1. Thus, holding $n_1 + n_2$ fixed and below Σm, minimizing $\underline{\lambda}(v, n)$ over n_1 and n_2 implies that $n_2 = 0$. If the window of price discovery for p is below $\Sigma m/N$, then we can ignore $n_1 + n_2$ above Σm.[12] Moreover, as long as the price impact vanishes and because the number of units K goes to infinity as the market grows large, δ goes to zero as

[12]In this case we have $p(n_1 + n_2) \geqslant \overline{v}$ for $n_1 + n_2 \geqslant \Sigma m$. Fixing $n_1 + n_2 = c \geqslant \Sigma m$ and minimizing $\underline{\lambda}(v, n)$ over n_1 and n_2 implies $n_1 = \Sigma m$, $n_2 = c - \Sigma n$ and $\underline{\lambda}(v, n) \geqslant \min(c, K)\overline{v} + \beta\Sigma m(r(c)(\overline{v} - v) - \delta) + \eta(c - \Sigma m)(r(c)\overline{v} - \delta) = \min(c, K)\overline{v} + \beta\min(c, K)\overline{v} - \beta\Sigma m r(c)v - \beta c\delta + (\eta - \beta)(c - \Sigma m)(r(c)\overline{v} - \delta)$, which is increasing in c if δ is sufficiently small.

well. Hence, as N goes large, $\underline{\lambda}(v,n)/K$ is converges to the strategic virtual objective in (3) with Σm playing the role of N. Theorem 5 then follows from the proof of Theorem 1.

7 Discussion

As we have said, our model has two interpretations: A monopolist with commitment power selling to a large market of buyers, or a decentralized market with demand uncertainty and complete information on the production side.

In the former interpretation, we have shown that market order mechanisms asymptotically extract all of the surplus, regardless of the information structure and equilibrium, as long as the seller uses pricing rules with low price impact and a narrow window of price discovery. Moreover, the achievable rate of $1/\sqrt{N}$ is known to be unimprovable in special cases. In fact, because of the simple binary-action structure of these mechanisms, we do not even need the full power of equilibrium. It would be enough to suppose that buyers prefer their strategies to the alternatives of always buying and never buying. And as the direct proof of Corollary 2 shows, if the low price is zero, then it is enough to suppose that buyers weakly prefer their strategies to never buying.[13] Our findings contrast sharply with those of the prior literature, which has largely focused on limit order mechanisms, such as first- or second-price auctions. As is well known, these mechanisms may admit equilibria that are far from competitive, even when the number of traders is large (Engelbrecht-Wiggans et al., 1983; Bergemann et al., 2017; Barelli et al., 2023).

Regarding the decentralized market, economists have long sought a tighter connection between large market models in which buyers are price takers with finite market models where individuals have small but non-negligible price impact. Of course, real markets feature complex dynamic feedback between orders and prices. The approach that is attempted in much of the literature, including this paper, is to reflect and approximate these rich dynamics with a static model, in which trading behavior is represented as a strategy in the normal form. In models with limit orders, the strategy is essentially a mapping from prices to price-contingent orders. This presumes that traders have access to all of the information that would be contained in the price, and it also aligns with the classical assumptions in rational expectations equilibrium. In contrast, we suppose that traders have access to *some*

[13]More broadly, the same argument would apply for any *coarse Bayes correlated equilibrium* in the market order mechanism, by which we mean any joint distribution over fundamentals and actions such that each player prefers their equilibrium strategy to any alternative strategy that always plays a fixed action. As Hartline, Syrgkanis, and Tardos (2015) show, no-regret learning dynamics are guaranteed to converge to a coarse Bayes correlated equilibrium in the long run. Thus, no-regret learning by buyers who participate in a large market order mechanism will necessarily lead to a competitive outcome.

information, which may or may not include all factors that determine the price, and they place their orders based on this information. This weaker informational assumption leads us to focus on equilibria of market order mechanisms. If the supply curve has small price impact and is close to inelastic, in the sense of having a narrow window of price discovery, then the equilibrium outcome in the decentralized market must be nearly efficient. And while inefficiency is possible in the elastic case, we also show that as long as price impact is small, welfare may be only a small amount below the no-information benchmark on a per capita basis. From a normative perspective, these are arguments in favor of practical implementation of market order mechanisms, and from a positive perspective, it provides new foundations for competitive behavior and efficiency in certain markets.

Why are market order mechanisms so robust to the details of information and equilibrium, whereas limit order mechanisms are not? A classical perspective is that the efficiency of markets under incomplete information depends on their ability to aggregate private information through prices. Moreover, for prices to aggregate private information, it seems that there should be relatively rich ways in which agents can interact with the market, e.g., a limit order that is rich enough to convey a buyer's expected value. Market order mechanisms, however, leave buyers with only the coarsest possible modes of interaction: buy or do not buy. Market order mechanisms therefore seem even less capable of aggregating information than limit order mechanisms. However, what may seem like a weakness is actually a strength: In settings where some agents have a large informational advantage, such as the proprietary information model of Engelbrecht-Wiggans, Milgrom, and Weber (1983), equilibrium in a limit order mechanism could be associated with a substantial winner's curse, because a single trader with an informational advantage can have a large effect on the terms of trade. But in a market order mechanism, traders are severely constrained in how they can leverage their private information, which in turn limits the scope for adverse selection. This is a key takeaway from our model: simple market mechanisms may limit information aggregation in a manner that reduces the scope for adverse selection, and thereby achieve superior welfare outcomes (cf. Bulow and Klemperer, 2002, for a related discussion and examples).

A substantive limitation of the current analysis is that the supply side is treated as exogenous. A natural direction for future work would be to consider two-sided markets, consisting of buyers and sellers, and where both sides must choose to participate in order for trade to take place. It is our hope that similar ideas can be used to construct market mechanisms will facilitate efficient trade in such settings.

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A Omitted proofs

Proof of Theorem 1. Fix an admissible low price $\underline{p}$. Let p be a pricing rule, $(\underline{p}, x, \epsilon)$ be a window of price discovery for p and γ the price impact of p.

Let[14]

$$\alpha = \max\left\{\frac{x}{1-x}, \frac{x\underline{v}}{\underline{v}-\underline{p}}\right\};$$
$$\beta = \alpha\frac{1-x}{x} - 1.$$

In the special case where $\underline{p} = \underline{v} = 0$, we set $\alpha = x/(1-x)$.

As in the proof of Corollary 2, we consider three cases.

Case 1: $n < N(x-\epsilon) - 1$, $nr(n) = \min\{K, n\} \leqslant \min\{K, Nx\}$, and hence

$$\frac{\lambda(v,n)}{\min\{K,Nx\}} \geqslant p(n+1)\frac{\min\{K,n\}}{\min\{K,Nx\}} + \frac{\alpha(N-n)r(n+1) - \beta nr(n)}{\min\{K,Nx\}}(v - p(n+1)) - \gamma(1+\beta).$$

Now, for any $p \leqslant \underline{p} \leqslant y\underline{v} \leqslant v$, where $y \in (0,1)$, consider the expression

$$\begin{aligned} f(n) &= p\min\{K,n\} + [\alpha(N-n)r(n+1) - \beta nr(n)](v-p) \\ &= p\min\{K,n\} + \left[\alpha\min\left\{K\frac{N-n}{n+1}, N-n\right\} - \beta\min\{K,n\}\right](v-p). \end{aligned}$$

The right-derivative with respect to n is

$$f'(n) = \begin{cases} p - (\alpha+\beta)(v-p) & \text{if } n+1 < K; \\ p - \left(\alpha\frac{K(N+1)}{(n+1)^2} + \beta\right)(v-p) & \text{if } n < K \leqslant n+1; \\ 0 - \alpha\frac{K(N+1)}{(n+1)^2}(v-p) & \text{if } n \geqslant K. \end{cases}$$

[14] Thus, the value of the multiplier α is tightly connected to the parameters of the pricing rule and the participation rates that minimize the lower bound on revenue. At first glance, this seems at odds with the analysis of Brooks and Du (2024), who emphasize that the Lagrange multiplier on obedience constraints corresponds to a choice of units for actions, and can be normalized to any value. But Brooks and Du (2024) analyze a limit of mechanisms in which the number of actions can be arbitrarily large, whereas market order mechanism has only two actions. When we constrain the number of actions in the mechanism, the nominal value of the Lagrange multiplier matters.

This expression is clearly non-positive if $n \geqslant K$. Note that

$$\alpha + \beta = \frac{\alpha}{x} - 1 \geqslant \frac{\underline{v}}{\underline{v} - \underline{p}} - 1$$

so that when $n + 1 < K$,

$$f'(n) = p - \left(\frac{\underline{v}}{\underline{v} - \underline{p}} - 1 \right) (\underline{v} - p) = \underline{v} - \underline{v} \frac{\underline{v} - p}{\underline{v} - \underline{p}} \leqslant 0.$$

Finally, when $n < K \leqslant n+1$, we have that when N is sufficiently large, $K(N+1)/(n+1)^2 \geqslant nNx/(n+1)^2 \geqslant n(n+3)/(n^2+2n+1) \geqslant 1$, so that $f'(n) \leqslant 0$ in this case as well. We conclude that

$$\begin{aligned}
\frac{\lambda(v,n)}{\min\{K, Nx\}} \geqslant p\,(n+1) \frac{\min\{K, Nx\}}{\min\{K, Nx\}} + \frac{\alpha(N - Nx)r(Nx+1) - \beta Nxr(Nx)}{\min\{K, Nx\}} (v - p\,(n+1)) \\
- \gamma\,(1+\beta) \\
= p\,(n+1) + \left(\alpha \frac{1-x}{x} \frac{\min\left\{K \frac{Nx}{Nx+1}, Nx\right\}}{\min\{K, Nx\}} - \beta \right) (v - p\,(n+1)) \\
- \gamma\,(1+\beta) \\
\geqslant p\,(n+1) + \left(\alpha \frac{1-x}{x} - \beta \right) (v - p\,(n+1)) \\
- \gamma\,(1+\beta) - \alpha \frac{1-x}{x} \left(1 - \frac{Nx}{Nx+1} \right) \overline{v} \\
\geqslant v - \gamma\,(1+\beta) - \alpha \frac{1-x}{x} \frac{1}{N(x+1)} \overline{v}.
\end{aligned}$$

We can then define $B_1 = (1+\beta)$ and

$$C_1 = \alpha \frac{1-x}{x(x+1)} \overline{v}.$$

Case 2: Now suppose $n > N(x+\epsilon)$. In this case, $p(n)$ is at least $\overline{v}$ at both n and $n+1$, and hence

$$\begin{aligned}
\lambda(v,n) \geqslant Nx \min\left\{ \frac{K}{Nx}, 1 \right\} p\,(n) + \alpha N(1-x) \min\left\{ \frac{K}{Nx+1}, 1 \right\} (v - p\,(n+1)) \\
- \beta \min\{K, Nx\}\,(v - p\,(n)) \\
\geqslant \min\{K, Nx\}\, p\,(n) + \alpha \frac{N(1-x)}{Nx+1} \min\{K, Nx+1\}\,(v - p\,(n) - \gamma) \\
- \beta \min\{K, Nx\}\,(v - p\,(n))
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \min\left\{K, Nx\right\} p\left(n\right) + \left[\alpha \frac{N(1-x)}{Nx+1} - \beta\right] \min\left\{K, Nx\right\} \left(v - p\left(n\right)\right) \\
&\quad - \alpha \frac{N(1-x)}{Nx+1} \left((\min\{K, Nx\} + 1)\gamma + \mathbb{I}_{K \geqslant Nx+1} \overline{v}\right) \\
&\geqslant \min\left\{K, Nx\right\} p\left(n\right) + \left[\alpha \frac{1-x}{x} - \beta\right] \min\left\{K, Nx\right\} \left(v - p\left(n\right)\right) \\
&\quad - \alpha \left|\frac{1-x}{x} - \frac{N(1-x)}{Nx+1}\right| \min\{K, Nx\} \overline{v} - \alpha \frac{N(1-x)}{Nx+1} \left((\min\{K, Nx\} + 1)\gamma + \mathbb{I}_{K \geqslant Nx+1} \overline{v}\right) \\
&\geqslant \min\{K, Nx\} \left[v - \alpha \left|\frac{1-x}{x} - \frac{N(1-x)}{Nx+1}\right| \overline{v} - \alpha \frac{N(1-x)}{Nx+1} \left(\gamma + \frac{\gamma + \mathbb{I}_{K \geqslant Nx+1} \overline{v}}{\min\{K, Nx\}}\right) \right] \\
&\geqslant \min\{K, Nx\} \left[v - \alpha \left|\frac{1-x}{x} - \frac{N(1-x)}{Nx+1}\right| \overline{v} - \alpha \frac{N(1-x)}{Nx+1} \left(2\gamma + \frac{\overline{v}}{Nx}\right) \right] \\
&\geqslant \min\{K, Nx\} \left[v - \alpha \frac{1-x}{x(Nx+1)} \overline{v} - \alpha \frac{1-x}{x} \left(2\gamma + \frac{\overline{v}}{Nx}\right) \right] \\
&\geqslant \min\{K, Nx\} \left[v - \alpha \frac{1-x}{x^2} \frac{1}{N} \overline{v} - \alpha \frac{1-x}{x} \left(2\gamma + \frac{\overline{v}}{Nx}\right) \right].
\end{aligned}
$$

We can then let

$$
B_2 = 2\alpha \frac{1-x}{x}
$$

and

$$
C_2 = 2\alpha \overline{v} \frac{1-x}{x^2}.
$$

Case 3: Finally, suppose $n \in [N(x-\epsilon), N(x+\epsilon)]$. Let us rewrite the strategic virtual objective as

$$
\begin{aligned}
\lambda(v, n) = v &\left[\alpha(N-n) \min\left\{\frac{K}{n+1}, 1\right\} - \beta \min\{K, n\}\right] \\
&+ \left(\min\{K, n\} - \left[\alpha(N-n) \min\left\{\frac{K}{n+1}, 1\right\} - \beta \min\{K, n\}\right]\right) p\left(n\right) \\
&- \alpha(N-n) \min\left\{\frac{K}{n+1}, 1\right\} \left(p\left(n+1\right) - p\left(n\right)\right) \\
\geqslant v &\left[\alpha \frac{1-x}{x} \min\left\{K \frac{N-n}{n+1} \frac{x}{1-x}, Nx \frac{N-n}{N(1-x)}\right\} - \beta \min\{K, n\}\right] \\
&- \left(\min\{K, n\} - \left[\alpha \frac{1-x}{x} \min\left\{K \frac{N-n}{n+1} \frac{x}{1-x}, Nx \frac{N-n}{N(1-x)}\right\} - \beta \min\{K, n\}\right]\right) \overline{v}
\end{aligned}
$$

$$
\begin{aligned}
&- \alpha\frac{1-x}{x}\min\left\{K\frac{N-n}{n+1}\frac{x}{1-x}, Nx\frac{N-n}{N(1-x)}\right\}\gamma \\
={}& v\min\{K,Nx\} - \left(\min\{K,Nx\} - \min\{K,n\}\right)v \\
&+ 2\alpha\frac{1-x}{x}\left[\min\left\{K\frac{N-n}{n+1}\frac{x}{1-x}, Nx\frac{N-n}{N(1-x)}\right\} - \min\{K,n\}\right]\overline{v} \\
&- \alpha\frac{1-x}{x}\min\left\{K\frac{N-n}{n+1}\frac{x}{1-x}, Nx\frac{N-n}{N(1-x)}\right\}\gamma \\
\geqslant{}& v\min\{K,Nx\} - \overline{v}\min\{K,Nx\}\frac{\epsilon}{x} \\
&+ 2\alpha\frac{1-x}{x}\left[\min\left\{K\frac{N-n}{n+1}\frac{x}{1-x}, Nx\frac{N-n}{N(1-x)}\right\} - \min\{K,Nx\}\left(1+\frac{\epsilon}{x}\right)\right]\overline{v} \\
&- \alpha\frac{1-x}{x}\min\left\{K\frac{N-n}{n+1}\frac{x}{1-x}, Nx\frac{N-n}{N(1-x)}\right\}\gamma.
\end{aligned}
$$

At this point, we need the inequalities

$$
\begin{aligned}
\frac{N-n}{n+1}\frac{x}{1-x} &\leqslant \frac{N-N(x-\epsilon)}{N(x-\epsilon)}\frac{x}{1-x} = \frac{1-x+\epsilon}{1-x}\frac{x}{x-\epsilon}; \\
\frac{N-n}{N(1-x)} &\leqslant \frac{N-N(x+\epsilon)}{N(1-x)} = \frac{1-x+\epsilon}{1-x} \leqslant \frac{1-x+\epsilon}{1-x}\frac{x}{x-\epsilon}; \\
\frac{N-n}{n+1}\frac{x}{1-x} &\geqslant \frac{N-Nx-N\epsilon}{N(x+\epsilon)+1}\frac{x}{1-x} = \frac{1-x-\epsilon}{1-x}\frac{x}{x+\epsilon+1/N}; \\
\frac{N-n}{N(1-x)} &\geqslant \frac{N-N(x+\epsilon)}{N(1-x)} = \frac{1-x-\epsilon}{1-x} \geqslant \frac{1-x-\epsilon}{1-x}\frac{x}{x+\epsilon+1/N}.
\end{aligned}
$$

Hence, assuming $\epsilon < x/2$,

$$
\begin{aligned}
\frac{\lambda(v,n)}{\min\{K,Nx\}} \geqslant{}& v - \overline{v}\frac{\epsilon}{x} + 2\alpha\frac{1-x}{x}\left[\frac{1-\frac{\epsilon}{1-x}}{1+\frac{\epsilon}{x}+\frac{1}{Nx}} - \left(1+\frac{\epsilon}{x}\right)\right]\overline{v} \\
&- \alpha\frac{1-x+\epsilon}{x-\epsilon}\gamma \\
={}& v - \overline{v}\frac{\epsilon}{x} - 2\alpha\frac{1-x}{x}\left[\frac{\frac{\epsilon}{x}+\frac{1}{Nx}+\frac{\epsilon}{1-x}}{1+\frac{\epsilon}{x}+\frac{1}{Nx}} + \frac{\epsilon}{x}\right]\overline{v} - \alpha\frac{1-x+\epsilon}{x-\epsilon}\gamma \\
\geqslant{}& v - \overline{v}\frac{\epsilon}{x} - 2\alpha\frac{1-x}{x}\left[\epsilon\left(\frac{2}{x}+\frac{1}{1-x}\right) + \frac{1}{Nx}\right]\overline{v} - \alpha\frac{2-x}{x}\gamma.
\end{aligned}
$$

Let

$$
\begin{aligned}
A &= \frac{\epsilon}{x} + 2\alpha\frac{1-x}{x}\left(\frac{2}{x}+\frac{1}{1-x}\right); \\
B_3 &= \alpha\frac{2-x}{x};
\end{aligned}
$$

$$C_3 = 2\alpha \frac{1-x}{x^2}.$$

Then taking $B = \max\{B_1, B_2, B_3\}$ and $C = \max\{C_1, C_2, C_3\}$ satisfies the hypotheses of the theorem. $\square$

Lemma 2. *Fix arbitrary $\alpha_i : \Omega_i \to \mathbb{R}_+$ and $\beta_i : \Omega_i \to \mathbb{R}_+$. We have the following lower bound on the revenue guarantee of Section 5.1:*

$$\underline{R}(p, \mu) \geqslant \int_{\nu,\omega} \min_a \lambda(\nu, \omega, a)\mu(d\nu, d\omega),$$

where

$$\lambda(\nu, \omega, a) = p(\Sigma a) \min\{K, \Sigma a\} + \sum_i \Big(\alpha_i(\omega_i)\mathbb{I}_{a_i=0} r(\Sigma a + 1)(v(\nu, \omega_i) - p(\Sigma a + 1)) \\ - \beta_i(\omega_i)\mathbb{I}_{a_i=1} r(\Sigma a)(v(\nu, \omega_i) - p(\Sigma a))\Big).$$

Likewise, we have the following lower bound on the welfare guarantee of Section 5.2:

$$\underline{W}(p, \mu) \geqslant \int_{\nu,\omega} \min_a \lambda(\nu, \omega, a)\mu(d\nu, d\omega),$$

where

$$\lambda(\nu, \omega, a) = \sum_i a_i v(\nu, \omega_i) - \sum_{m \leqslant n} p(m) + \sum_i \Big(\alpha_i(\omega_i)\mathbb{I}_{a_i=0}(v(\nu, \omega_i) - p(\Sigma a + 1)) \\ - \beta_i(\omega_i)\mathbb{I}_{a_i=1}(v(\nu, \omega_i) - p(\Sigma a))\Big).$$

Proof of Lemma 2. Let us focus on the first part; the proof for the second part is analogous.

Any equilibrium on an information structure induces an outcome $\sigma \in \Delta(\mathcal{V} \times \Omega \times A)$ whose marginal is μ ($\mathrm{marg}_{\mathcal{V}\times\Omega}\, \sigma = \mu$), and the obedience constraints hold for all i and ω_i:

$$\sum_{a_{-i},\omega_{-i},\nu} r(1 + \Sigma a_{-i})(v(\nu, \omega_i) - p(1 + \Sigma a_{-i}))\sigma(\nu, \omega, (0, a_{-i})) \leqslant 0,$$
$$-\sum_{a_{-i},\omega_{-i},\nu} r(1 + \Sigma a_{-i})(v(\nu, \omega_i) - p(1 + \Sigma a_{-i}))\sigma(\nu, \omega, (1, a_{-i})) \leqslant 0.$$

Therefore, we have

$$
\begin{aligned}
&\sum_{a,\omega,\nu} \min\{K, \Sigma a\} p(\Sigma a)\sigma(\nu,\omega,a) \\
&\geqslant \sum_{a,\omega,\nu} \min\{K, \Sigma a\} p(\Sigma a)\sigma(\nu,\omega,a) \\
&\quad + \sum_i \sum_{\omega_i} \sum_{a_{-i},\omega_{-i},\nu} \alpha_i(\omega_i) r(1+\Sigma a_{-i})(v(\nu,\omega_i) - p(1+\Sigma a_{-i}))\sigma(\nu,\omega,(0,a_{-i})) \\
&\quad - \sum_i \sum_{\omega_i} \sum_{a_{-i},\omega_{-i},\nu} \beta_i(\omega_i) r(1+\Sigma a_{-i})(v(\nu,\omega_i) - p(1+\Sigma a_{-i}))\sigma(\nu,\omega,(1,a_{-i})) \\
&= \sum_{a,\omega,\nu} \lambda(\nu,\omega,a)\sigma(\nu,\omega,a) \\
&\geqslant \sum_{\omega,\nu} \min_a \lambda(\nu,\omega,a)\mu(\nu,\omega).
\end{aligned}
$$

□

Proof of Theorem 3. Let (p_N) be a sequence of pricing rules with corresponding windows of price discovery $(0, x_N, \epsilon_N)$ and price impacts γ_N, where $x_N \to x \in (0,1)$, $\epsilon_N \to 0$ and $\gamma_N \to 0$. And let (δ_N) be a sequence converging to zero for which condition (8) holds.

Let $y = F^{-1}(1-x)$, and let $y^- = \max\{z \in \bigcup_i \Omega_i : z < y\}$; if $y = \min \bigcup_i \Omega_i$, then set $y^- = y - 1$. Notice that by definition we have $F(y^-) < 1 - x$. Set

$$
\alpha_{N,i}(\omega_i) = \begin{cases} \frac{x_N}{1-F(y^-)+\delta_N - x_N} & \omega_i \geqslant y, \\ 0 & \omega_i < y, \end{cases} \qquad \beta_{N,i}(\omega_i) = \begin{cases} 0 & \omega_i \geqslant y, \\ \frac{x_N}{1-F(y^-)+\delta_N - x_N} & \omega_i < y. \end{cases}
$$

We apply the first part of Lemma 2 to each p_N with the above multipliers, which yields

$$
\underline{R}(p_N, \mu_N) \geqslant \int_{\nu,\omega} \min_a \lambda_N(\nu,\omega,a)\mu_N(d\nu, d\omega),
$$

where

$$
\begin{aligned}
\lambda_N(\nu,\omega,a) \geqslant{}& \min\{K_N, n_0+n_1\} p_N(n_0+n_1) \\
&+ \frac{x_N}{1-F(y^-)+\delta_N - x_N}(N_1(\omega) - n_1) r_N(n_0+n_1+1)(v(\nu,y) - p_N(n_0+n_1+1)) \\
&- \frac{x_N}{1-F(y^-)+\delta_N - x_N} n_0 r_N(n_0+n_1)(v(\nu,y^-) - p_N(n_0+n_1)) \\
\equiv{}& \underline{\lambda}_N(\nu, N_1(\omega), n_0, n_1),
\end{aligned}
$$

where $N_1(\omega) = |\{i : \omega_i \geqslant y\}|$, $n_1 = |\{i : \omega_i \geqslant y, a_i = 1\}|$, $n_0 = |\{i : \omega_i < y, a_i = 1\}|$. Therefore,

$$\min_a \lambda_N(\nu, \omega, a) \geqslant \min_{n_1 \leqslant N_1(\omega), n_0 \leqslant N - N_1(\omega)} \underline{\lambda}_N(\nu, N_1(\omega), n_0, n_1).$$

We have $r_N(n_0+n_1+1) - r_N(n_0+n_1) = 0$ if $n_0+n_1+1 \leqslant K_N$ and $|r_N(n_0+n_1+1) - r_N(n_0+n_1)| = \frac{K_N}{(n_0+n_1)(n_0+n_1+1)} \leqslant \frac{1}{K_N-1}$ if $n_0+n_1+1 > K_N$. Moreover $|p_N(n_0+n_1+1) - p_N(n_0+n_1)| \leqslant \gamma_N$. Since both $\frac{1}{K_N-1}$ and γ_N tend to zero as $N \to \infty$ and $v(\nu, y) > v(\nu, y^-)$, when N is sufficiently large, we have

$$r_N(n_0+n_1+1)(v(\nu, y) - p_N(n_0+n_1+1)) > r_N(n_0+n_1)(v(\nu, y^-) - p_N(n_0+n_1)),$$

which implies that

$$\underline{\lambda}_N(\nu, N_1(\omega), n_0, n_1) \geqslant \underline{\lambda}_N(\nu, N_1(\omega), 0, n_0+n_1) \tag{15}$$

if $n_0 + n_1 \leqslant N_1(\omega)$, and

$$\underline{\lambda}_N(\nu, N_1(\omega), n_0, n_1) \geqslant \underline{\lambda}_N(\nu, N_1(\omega), n_0+n_1-N_1(\omega), N_1(\omega)) \tag{16}$$

if $n_0 + n_1 > N_1(\omega)$.

Suppose N is sufficiently large so that

$$x_N + \epsilon_N < 1 - F(y^-) - \delta_N.$$

We will focus on ω such that

$$(1 - F(y^-) - \delta_N)N \leqslant N_1(\omega) \leqslant (1 - F(y^-) + \delta_N)N. \tag{17}$$

Since (p_N) is bounded, there exists a constant $C > 0$ such that $|\underline{\lambda}_N(\nu, N_1(\omega), n_0, n_1)| \leqslant CN$ for all n_0 n_1, and N. Let Ω' be the set of ω for which (17) does not hold. Then,

$$\int_{(\nu,\omega)\in\mathcal{V}\times\Omega'} \left|\min_{n_0,n_1} \underline{\lambda}_N(\nu, N_1(\omega), n_0, n_1)\right| \mu_N(d\nu, d\omega) \leqslant CN\mu_N\left(\sup_{z\in\mathbb{R}} |F_N(z) - F(z)| > \delta_N\right)$$

so the ratio of the above to $\min\{K_N, Nx\}$ tends to zero as $N \to \infty$ by (8).

Set

$$n = n_0 + n_1.$$

As in the proof of Theorem 1 we consider three cases:

Case 1: $n < N(x_N - \epsilon_N) - 1$. In this case (15) applies, and

$$\begin{aligned}
&\underline{\lambda}_N(\nu, N_1(\omega), n_0, n_1) \\
\geqslant &\underline{\lambda}_N(\nu, N_1(\omega), 0, n) \\
= &\frac{x_N}{1 - F(y^-) + \delta_N - x_N}(N_1(\omega) - n) r_N(n+1) v(\nu, y),
\end{aligned}$$

which is clearly decreasing in n, and so is at least

$$\frac{x_N}{1 - F(y^-) + \delta_N - x_N}(N_1(\omega) - Nx) r_N(Nx+1) v(\nu, y) \tag{18}$$

Case 2: $n > N(x_N + \epsilon_N)$.

Subcase a: $n \leqslant N_1(\omega)$. Then (15) applies:

$$\begin{aligned}
&\underline{\lambda}_N(\nu, N_1(\omega), n_0, n_1) \\
\geqslant &\underline{\lambda}_N(\nu, N_1(\omega), 0, n) \\
= &\min\{K_N, n\} p_N(n) + \frac{x_N}{1 - F(y^-) + \delta_N - x_N}(N_1(\omega) - n) r_N(n+1)(v(\nu, y) - p_N(n+1)) \\
\geqslant &\min\{K_N, n\} p_N(n) + \frac{x_N}{1 - F(y^-) + \delta_N - x_N}(N_1(\omega) - n) r_N(n+1)(v(\nu, y) - \gamma_N - p_N(n)).
\end{aligned}$$

Examining the coefficients of $p_N(n)$, we note that $\min\{K_N, n\} = n r_N(n) \geqslant \frac{x_N}{1-F(y^-)+\delta_N - x_N}(N_1(\omega) - n) r_N(n+1)$. Since $p_N(n) \geqslant \overline{v}$ in this case, the last line above is at least

$$\begin{aligned}
&\min\{K_N, n\}\overline{v} + \frac{x_N}{1 - F(y^-) + \delta_N - x_N}(N_1(\omega) - n) r_N(n+1)(v(\nu, y) - \gamma_N - \overline{v}) \\
\geqslant &\min\{K_N, Nx\}\overline{v} + \frac{x_N}{1 - F(y^-) + \delta_N - x_N}(N_1(\omega) - Nx) r_N(Nx+1)(v(\nu, y) - \gamma_N - \overline{v}),
\end{aligned} \tag{19}$$

since the left-hand side is increasing in n.

Subcase b: $n > N_1(\omega)$. Then (16) applies:

$$\begin{aligned}
&\underline{\lambda}_N(\nu, N_1(\omega), n_0, n_1) \\
\geqslant &\underline{\lambda}_N(\nu, N_1(\omega), n - N_1(\omega), N_1(\omega)) \\
= &\min\{K_N, n\} p_N(n) - \frac{x_N}{1 - F(y^-) + \delta_N - x_N}(n - N_1(\omega)) r_N(n)(v(\nu, y^-) - p_N(n)) \\
\geqslant &\min\{K_N, n\}\overline{v} \\
\geqslant &\min\{K_N, Nx\}\overline{v}
\end{aligned} \tag{20}$$

since $p_N(n) \geqslant \overline{v}$ and $n \geqslant Nx$ in this case.

Case 3: $n \in [N(x_N - \epsilon_N) - 1, N(x_N + \epsilon_N)]$. In this case (15) applies:

$$\begin{aligned}
&\underline{\lambda}_N(\nu, N_1(\omega), n_0, n_1) \\
\geqslant &\underline{\lambda}_N(\nu, N_1(\omega), 0, n) \\
= &\min\{K_N, n\}p_N(n) + \frac{x_N}{1 - F(y^-) + \delta_N - x_N}(N_1(\omega) - n)r_N(n+1)(v(\nu, y) - p_N(n+1)) \\
\geqslant &\min\{K_N, n\}p_N(n) + \frac{x_N}{1 - F(y^-) + \delta_N - x_N}(N_1(\omega) - n)r_N(n+1)(v(\nu, y) - \gamma_N - p_N(n))
\end{aligned} \tag{21}$$

Examining (18), (19), (20), and (21) and applying condition (8) on $N_1(\omega)$ and the condition of case 3 on n, we see that

$$\liminf_{N\to\infty} \left(\frac{\int_{\nu,\omega} \min_a \lambda_N(\nu, \omega, a)\mu_N(d\nu, d\omega)}{\min\{K_N, Nx\}} - \int_{\nu,\omega} v(\nu, y)\mu_N(d\nu, d\omega) \right) \geqslant 0.$$

□

Proof of Theorem 4. Let (μ_N) be a sequence of priors satisfying condition (12) for some sequence (δ_N) converging to zero, and let (p_N) be a sequence of supply functions with price impact γ_N tending to zero.

Let $f_N(y) = \{i : \omega_i = y\}/N$, i.e., the empirical probability mass function for the private values. Likewise, let $f(y)$ be the probability mass function corresponding to the limit CDF $F(y)$, i.e., $F(y) = \sum_{z \leqslant y} f(z)$, where the sum are over $z \in \bigcup_i \Omega_i$.

Because of (12), we can assume

$$\sup_y |F_N(y) - F(y)| \leqslant \delta_N, \tag{22}$$

which implies

$$\sup_y |f_N(y) - f(y)| \leqslant 2\delta_N.$$

Let n_N^* be the maximizer for $W^*(\mu_N)$ in (11). Let $y_N^* = F^{-1}(1 - n_N^*/N)$. By definition, we have $F(y_N^*) \geqslant 1 - n_N^*/N$.

We set

$$\alpha_{N,i}(\omega_i) = \mathbb{I}_{\omega_i > y_N^*} + \frac{n_N^*/N - (1 - F(y_N^*))}{f(y_N^*)}\mathbb{I}_{\omega_i = y_N^*}, \quad \beta_{N,i}(\omega_i) = 1 - \alpha_{N,i}(\omega_i).$$

Define the strategic virtual objective from the second part of Lemma 2 with above multipliers:

$$\lambda_N(\nu,\omega,a) = \sum_i a_i v(\nu,\omega_i) - \sum_{m\leqslant \Sigma a} p_N(m) + \sum_i \Big(\mathbb{I}_{a_i=0}\alpha_{N,i}(\omega_i)(v(\nu,\omega_i) - p_N(\Sigma a+1)) \\ - \mathbb{I}_{a_i=1}\beta_{N,i}(\omega_i)(v(\nu,\omega_i) - p_N(\Sigma a))\Big).$$

We have, for any a,

$$\begin{aligned}\lambda_N(\nu,\omega,a) &\geqslant \sum_i a_i v(\nu,\omega_i) - \sum_{m\leqslant \Sigma a} p_N(m) + \sum_i \Big(\mathbb{I}_{a_i=0}\alpha_{N,i}(\omega_i)(v(\nu,\omega_i) - p_N(\Sigma a)) \\ &\qquad - \mathbb{I}_{a_i=1}\beta_{N,i}(\omega_i)(v(\nu,\omega_i) - p_N(\Sigma a))\Big) - N\gamma_N \\ &= \Sigma a p_N(\Sigma a) - \sum_{m\leqslant \Sigma a} p_N(m) + \sum_i \Big(\mathbb{I}_{a_i=0}\alpha_{N,i}(\omega_i)(v(\nu,\omega_i) - p_N(\Sigma a)) \\ &\qquad + \mathbb{I}_{a_i=1}\alpha_{N,i}(\omega_i)(v(\nu,\omega_i) - p_N(\Sigma a))\Big) - N\gamma_N \\ &= \Sigma a p_N(\Sigma a) - \sum_{m\leqslant \Sigma a} p_N(m) + \sum_{y>y_N^*} N f_N(y)(v(\nu,y) - p_N(\Sigma a)) \\ &\qquad + \frac{n_N^*/N - (1-F(y_N^*))}{f(y_N^*)} N f_N(y_N^*)(v(\nu,y_N^*) - p_N(\Sigma a)) - N\gamma_N.\end{aligned}$$

Under condition (22), the above is at least

$$\begin{aligned}&\Sigma a p_N(\Sigma a) - \sum_{m\leqslant \Sigma a} p_N(m) + \sum_{y>y_N^*} N f(y)(v(\nu,y) - p_N(\Sigma a)) \\ &\quad + \frac{n_N^*/N - (1-F(y_N^*))}{f(y_N^*)} N f(y_N^*)(v(\nu,y_N^*) - p_N(\Sigma a)) - N\gamma_N - N\delta_N C \\ &= (\Sigma a - n_N^*) p_N(\Sigma a) - \sum_{m\leqslant \Sigma a} p_N(m) + \sum_{y>y_N^*} N f(y) v(\nu,y) + (n_N^* - N(1-F(y_N^*)))v(\nu,y_N^*) \\ &\quad - N\gamma_N - N\delta_N C \\ &\geqslant -\sum_{m\leqslant n_N^*} p_N(m) + \sum_{y>y_N^*} N f(y) v(\nu,y) + (n_N^* - N(1-F(y_N^*)))v(\nu,y_N^*) - N\gamma_N - N\delta_N C\end{aligned} \tag{23}$$

where for the last inequality we used the same reasoning with the J function as in the proof of Theorem 2, and

$$C = 2(\overline{v}+\overline{p})\left|\bigcup_i \Omega_i\right|, \qquad \overline{p} = \sup_{N,n} p_N(n).$$

Let $y_N = F_N^{-1}(1 - n_N^*/N)$, then

$$\sum_{m=1}^{n_N^*} v(\nu, \omega^{(m)}) = \sum_{y > y_N} N f_N(y) v(\nu, y) + (n_N^* - N(1 - F_N(y_N))) v(\nu, y_N),$$

where $\omega^{(m)}$ is the m-th highest private value among ω, and under condition (22), we have

$$\Bigg| \left(\sum_{y > y_N} N f_N(y) v(\nu, y) + (n_N^* - N(1 - F_N(y_N))) v(\nu, y_N) \right) \\ - \left(\sum_{y > y_N^*} N f(y) v(\nu, y) + (n_N^* - N(1 - F(y_N^*))) v(\nu, y_N^*) \right) \Bigg| \leqslant N \delta_N C.$$

Combining the above with (23), the theorem follows from Lemma 2. □

B Uncertain number of buyers and goods

In this section, we enrich the common value model with uncertain numbers of buyers and goods. Let the state space be $\Theta = [0,1] \times \mathbb{Z}_+ \times \mathbb{Z}_+$. A state $\theta = (v, N, K) \in \Theta$ means that there are K units of a good, and there are N buyers with a unit-demand and a pure common value v for the good. Let $\mu \in \Delta(\Theta)$ be a distribution over the states. We suppose $\mu(\{N \leqslant \overline{N}\}) = 1$ for some $\overline{N} \in \mathbb{Z}_+$.[15] Uncertainty in the number of buyers is a common feature in markets where agents trade through an online platform; moreover, the number of buyers could be correlated with the value because the auctioneer solicits the participants in the auction after learning some information about the value (Lauermann and Wolinsky, 2017, 2022; Lauermann and Speit, 2023). Likewise, the number of goods could be uncertain and correlated with the value, for example, because some units of goods are reserved for some "non-competitive" investors in the treasury auction.

The buyers' private information about the state is described by an information structure $I = (S, \sigma)$ as in Section 2, where S_i is a finite set of signals (or types) for buyer i, $S = \prod_{i=1}^{\overline{N}} S_i$, and $\sigma \in \Delta(\Theta \times S)$ is the joint distribution of the states and signals such that $\mathrm{marg}_\Theta\, \sigma = \mu$. Moreover, we require each S_i contains a null type $\varnothing$; if $s_i = \varnothing$, then buyer i is not present. Thus, for consistency we also require that for every (v, N, K, s) in the support of σ, we have $N = |\{i : s_i \neq \varnothing\}|$.

[15] In Theorem 6 below, we will consider a sequence of priors μ_l where the number of buyers goes to infinity in probability, and the corresponding upper bounds $\overline{N}_l$ go to infinity as well.

A market order mechanism is defined as in Section 2. To model the absence of some buyers, for every i we now add a null action $\varnothing$ to the action space: $A_i = \{\varnothing, 0, 1\}$. The rationing probability is $r(a) = \min\{K/n(a), 0\}$, and pricing function $p(x)$ depends only on the participation rate $x = n(a)/N(a)$, where $n(a) = |\{i : a_i = 1\}|$ is the aggregate order, and $N(a) = |\{i : a_i \neq \varnothing\}|$ is the potential number of buyers. A buyer i who is present in the mechanism ($a_i \neq \varnothing$) has utility $u_i = a_i r(a)(v - p(x))$ as before.

For a strategy $b_i : S_i \to \Delta(A_i)$, we now require $b_i(s_i) = \varnothing$ for the null type $s_i = \varnothing$. Subject to this constraint on the strategy, the definitions of equilibrium and revenue guarantee remain the same as before.

For simplicity, let us focus on piecewise linear pricing rule $p_{\widehat{x},\epsilon}(x)$ with a fixed low price $\underline{p} < \underline{v}$:

$$p_{\widehat{x},\epsilon}(x) = \begin{cases} \underline{p} & x < \widehat{x} - \epsilon, \\ \underline{p} + (\overline{v} - \underline{p})\frac{x - (\widehat{x} - \epsilon)}{2\epsilon} & x \in [\widehat{x} - \epsilon, \widehat{x} + \epsilon], \\ \overline{v} & x > \widehat{x} + \epsilon. \end{cases}$$

Theorem 6. *Let (μ_l) be a sequence of state distributions and let (v_l, N_l, K_l) be the corresponding sequence of random variables. Suppose ϵ_l converges to 0 and $\epsilon_l N_l$ converges in probability to ∞ as $l \to \infty$. Then there exists a sequence of random variables δ_l such that $\underline{R}(p_{\widehat{x},\epsilon_l}, \mu_l) \geqslant \mathbb{E}_{\mu_l}[\min\{K_l, \widehat{x} N_L\}(v_l - \delta_l)]$ for every l, and δ_l converges in probability to zero as $l \to \infty$.*

The proof follows immediately from Corollary 2: we set $\delta_l = A\epsilon_l + \frac{B}{N\epsilon_l} + \frac{C}{N_l}$, where the constants A, B and C are from the corollary.

Thus, the equilibrium price in the piecewise-linear market order mechanism is guaranteed to converge to the value in expectation, regardless of the correlation between the value and the number of buyers. In contrast, Lauermann and Wolinsky (2017, 2022) show that a common value first price auction generally have low price equilibrium (bounded away from the expected value) even as the number of buyers converges to infinity in probability, as long as there are relatively more buyers given a low value than a high value.